ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to Fibonacci@MathPro.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEM PROPOSED IN THIS ISSUE

B-820 ***Proposed by the editor; dedicated to Herta T. Freitag***

Find a recurrence (other than the usual one) that generates the Fibonacci sequence.

[The usual recurrence is a second-order linear recurrence with constant coefficients. Can you find a first-order recurrence that generates the Fibonacci sequence? Can you find a third-order linear recurrence? a nonlinear recurrence? one with nonconstant coefficients? etc.]

SOLUTIONS

A Disguise for Zero

B-795 ***Proposed by Wray Brady, Jalisco, Mexico***
(Vol. 33, no. 4, August 1995)

Evaluate

$$E = \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!} L_{2n}.$$

Solution by L. A. G. Dresel, Reading, England

Since $L_n = \alpha^n + \beta^n$, the given expression is

$$E = \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!} (\alpha^n + \beta^n).$$

Comparing this with the well-known power series for $\cos(x)$, we have $E = \cos(\pi\alpha) + \cos(\pi\beta)$. Since $\alpha + \beta = 1$, we have $\cos\pi\alpha = \cos(\pi - \pi\beta) = -\cos(\pi\beta)$, giving $E = 0$.

Seiffert showed that $\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n}}{(2n)!} L_{2kn} = 0$ *if and only if k is not divisible by 3.*

Also solved by Glenn Bookhout, Paul S. Bruckman, Charles K. Cook, Andrej Dujella, Russell Euler, C. Georghiou, Russell Jay Hendel, Hans Kappus, Joseph J. Koštál, Can. A. Minh, Bob Prielipp, R. P. Sealy, H.-J. Seiffert, Sahib Singh, and the proposer.

A Disguise for Five

B-796 ***Proposed by M. N. S. Swamy, St. Lambert, Quebec, Canada (Vol. 33, no. 5, November 1995)***

Show that

$$\frac{L_n^2 + L_{n+1}^2 + L_{n+2}^2 + \cdots + L_{n+a}^2}{F_n^2 + F_{n+1}^2 + F_{n+2}^2 + \cdots + F_{n+a}^2}$$

is always an integer if a is odd.

Solution by Sahib Singh, Clarion University of Pennsylvania, PA

We prove that the value of the given expression is 5. The result follows from identity (I_{12}) of [1]: $L_n^2 = 5F_n^2 + 4(-1)^n$. Using this identity yields

$$L_n^2 + L_{n+1}^2 = 5(F_n^2 + F_{n+1}^2),$$
$$L_{n+2}^2 + L_{n+3}^2 = 5(F_{n+2}^2 + F_{n+3}^2),$$
$$\cdots$$
$$L_{n+a-1}^2 + L_{n+a}^2 = 5(F_{n+a-1}^2 + F_{n+a}^2).$$

Addition yields $L_n^2 + L_{n+1}^2 + \cdots + L_{n+a}^2 = 5(F_n^2 + F_{n+1}^2 + \cdots + F_{n+a}^2)$, from which the result follows.

Reference

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, Calif.: The Fibonacci Association, 1979.

Also solved by Charles Ashbacher, Wray Brady, Paul S. Bruckman, Andrej Dujella, Russell Euler, Herta T. Freitag, Russell Jay Hendel, Joseph J. Koštál, Carl Libis, Can. A. Minh, Bob Prielipp, H.-J. Seiffert, and the proposer.

Decimal Congruence

B-797 ***Proposed by Andrew Cusumano, Great Neck, NY (Vol. 33, no. 5, November 1995)***

Let $\langle H_n \rangle$ be any sequence that satisfies the recurrence $H_{n+2} = H_{n+1} + H_n$. Prove that $7H_n \equiv H_{n+15} \pmod{10}$

Solution by Russell Euler, Northwest Missouri State University, Maryville, MO

From formula (8) of [1], we have $H_{n+15} = F_{14}H_n + F_{15}H_{n+1}$. So $H_{n+15} = 377H_n + 610H_{n+1} \equiv 7H_n \pmod{10}$.

Reference

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications.* Chichester: Ellis Horwood Ltd., 1989.

Haukkanen showed how to generate many sets of integers c, k, and m, such that $cH_n \equiv H_{n+k} \pmod{m}$ *for all n.*

Also solved by Charles Ashbacher, Brian D. Beasley, David M. Bloom, Wray Brady, Paul S. Bruckman, Andrej Dujella, Herta T. Freitag, Pentti Haukkanen, Russell Jay Hendel, Gerald A. Heuer, Joseph J. Koštál, Carl Libis, Can. A. Minh, Bob Prielipp, R. P. Sealy, H.-J. Seiffert, Sahib Singh, and the proposer.

Powers of 5

B-798 ***Proposed by Seung-Jin Bang, Ajou University, Suwon, Korea (Vol. 33, no. 5, November 1995)***

Prove that, for n a positive integer, F_{5^n} is divisible by 5^n but not by 5^{n+1}.

Comment by the editor: Several readers pointed out that this is a duplicate of problem B-248. Sorry about that. How could I have missed this? See this Quarterly (1973)553 for the solution.

Bloom mentioned the stronger result that, if p is odd and $k \geq 1$, *then* F_{np} *is divisible by* p^{k+1} *but not* p^{k+2}. *This was proven by Lucas in 1876; see page 396 in [1].*

Reference

1. L. E. Dickson. *History of the Theory of Numbers.* Vol. 1. New York: Chelsea, 1971.

Also proposed by V. E. Hoggatt, Jr. Also solved by Brian D. Beasley, David M. Bloom, Wray Brady, Paul S. Bruckman, Warren Cheeves, Andrej Dujella, Russell Euler, Herta T. Freitag, Pentti Haukkanen, Can. A. Minh, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Gregory Wulczyn, and the proposers.

A Recurrence

B-799 ***Proposed by David Zeitlin, Minneapolis, MN (Vol. 33, no. 5, November 1995)***

Solve the recurrence $A_{n+2} = 4A_{n+1} + A_n$, for $n \geq 0$, with initial conditions $A_0 = 1$ and $A_1 = 4$; expressing your answer in terms of Fibonacci and/or Lucas numbers.

Solution by David M. Bloom, Brooklyn College, NY

From formula (15a) of [1], we have that, for all a and b, $F_{a+b} + (-1)^b F_{a-b} = F_a L_b$. Setting $b = 3$ gives $F_{a+3} - F_{a-3} = 4F_a$. Hence, if we let $A_n = \frac{1}{2}(F_{3n+1} + F_{3n+2})$, we have $A_{n+1} - A_{n-1} = \frac{1}{2}(F_{3n+4} + F_{3n+5} - F_{3n-2} - F_{3n-1}) = \frac{1}{2}(4F_{3n+1} + 4F_{3n+2}) = 4A_n$. Clearly, $A_0 = 1$ and $A_1 = 4$, so these A's coincide with the A's of the problem. Hence, $A_n = \frac{1}{2}(F_{3n+1} + F_{3n+2}) = \frac{1}{2}F_{3n+3}$.

Reference

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications.* Chichester: Ellis Horwood Ltd., 1989.

Seiffert found the generalization that, if k is a positive integer, and the sequence $A_n(k)$ satisfies the recurrence $A_{n+2}(k) = L_k A_{n+1}(k) - (-1)^k A_n(k)$, for $n \geq 0$, with initial conditions $A_0(k) = 1$ and $A_1(k) = L_k$, then we have $A_n(k) = F_{nk+k} / F_k$. This is an immediate consequence of the result of problem B-748, see this Quarterly 33.1(1995):88.

Also solved by Brian D. Beasley, Paul S. Bruckman, Andrej Dujella, Russell Euler, Herta T. Freitag, Pentti Haukkanen, Russell Jay Hendel, Joseph J. Koštál, Daina A. Krigens, Carl Libis, Graham Lord, Can. A. Minh, Bob Prielipp, V. Ravoson & R. Caboz (jointly), H.-J. Seiffert, Sahib Singh, M. N. S. Swamy, and the proposer.

Pell/Fibonacci Inequality

B-800 *Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 33, no. 5, November 1995)*

Define the Pell numbers by the recurrence $P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$, with initial conditions $P_0 = 0$ and $P_1 = 1$.

Show that, for all integers $n \geq 4$, $P_n < F_{k(n)}$, where $k(n) = \lfloor (11n+2)/6 \rfloor$.

Solution by Paul S. Bruckman, Highwood, IL

Given $u = 1 + \sqrt{2}$, we may easily verify that $u^6 = 70u + 29$ and $u^{12} = 13860u + 5741$; hence, $u^{12} = 198u^6 - 1$. From this we may deduce that $P_{n+12} = 198P_{n+6} - P_n$ for all integers n. Likewise, it is straightforward to show that $F_{n+22} = 199F_{n+11} + F_n$.

Our next observation is the following:

$$k(n+6) = k(n) + 11.$$

We will use these identities to establish the desired result by induction.

Let S denote the set of integers $n \geq 4$ for which $P_n < F_{k(n)}$. Our first step is to construct a table of the first few values of P_n and $F_{k(n)}$. These are given below.

n	$k(n)$	P_n	$F_{k(n)}$
4	7	12	13
5	9	29	34
6	11	70	89
7	13	169	233
8	15	408	610
9	16	985	987
10	18	2378	2584
11	20	5741	6765
12	22	13860	17711
13	24	33461	46368
14	26	80782	121393
15	27	195025	196418

Note that $P_3 = 5 = F_5 = F_{k(3)}$, justifying the restriction $n \geq 4$. We see from the table that $n \in S$ for $4 \leq n \leq 15$. Assume that $n \in S$ and $(n+6) \in S$. Then $P_n < F_{k(n)}$ and $P_{n+6} < F_{k(n+6)} = F_{k(n)+11}$. Then $P_{n+12} = 198P_{n+6} - P_n < 198F_{k(n)+11} < 199F_{k(n)+11} + F_{k(n)} = F_{k(n)+22}$, or $P_{n+12} < F_{k(n+12)}$. Thus, $n \in S$ and $(n+6) \in S$ implies that $(n+12) \in S$, and the proof by induction is complete.

Also solved by Andrej Dujella, Russell Jay Hendel, and the proposer. One incorrect solution was received.

Congruences mod 40

B-801 ***Proposed by Larry Taylor, Rego Park, NY***
(Vol. 33, no. 5, November 1995)

Let $k \geq 2$ be an integer and let n be an odd integer. Prove that

(a) $F_{2^k} \equiv 27 \cdot 7^k \pmod{40}$;

(b) $F_{n2^k} \equiv 7^k F_{16n} \pmod{40}$.

Solution by H.-J. Seiffert, Berlin, Germany

(a) From [2], we know that

$$L_{2^j} \equiv 7 \pmod{40}, \text{ for } j \geq 2. \tag{1}$$

Repeated application of the equation (I_7) of [1], $F_{2m} = F_m L_m$, gives

$$F_{2^k} = F_4 \prod_{j=2}^{k-1} L_{2^j}, \quad k \geq 2,$$

so that, by $F_4 = 3$ and (1), we have $F_{2^k} \equiv 3 \cdot 7^{k-2} \equiv 27 \cdot 7^k \pmod{40}$, for $k \geq 2$.

(b) Let $k \geq 2$. We shall prove that

$$F_{n2^k} \equiv 7^k F_{16n} \pmod{40}, \text{ for all integers } n. \tag{2}$$

It suffices to prove (2) for $n \geq 0$, since $F_{-2m} = -F_{2m}$. Since $F_{16} = 987 \equiv 27 \pmod{40}$, (2) is true for $n = 1$, by part (a). Clearly, it is also true for $n = 0$. Suppose that (2) holds for all $j \in \{0, 1, 2, \ldots, n\}$, $n \geq 1$. Then, by equation (I_{21}) of [1] and (1),

$$\begin{aligned} F_{(n+1)2^k} &= L_{2^k} F_{n2^k} - F_{(n-1)2^k} \equiv 7 \cdot 7^k F_{16n} - 7^k F_{16(n-1)} \\ &= 7^k (7F_{16n} - F_{16(n-1)}) \equiv 7^k (L_{16} F_{16n} - F_{16(n-1)}) \\ &= 7^k F_{16(n+1)} \pmod{40}. \end{aligned}$$

This completes the induction proof of (2).

References

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Santa Clara, Calif.: The Fibonacci Association, 1979.
2. S. Singh. Problem B-694. *The Fibonacci Quarterly* **30.3** (1992):276.

Also solved by Paul S. Bruckman, Andrej Dujella, Russell Jay Hendel, Joseph J. Koštál, and Bob Prielipp.

NOTE: The Elementary Problems Column is in need of more *easy*, yet elegant and non-routine problems.