# A SIMPLER GRAMMAR FOR FIBONACCI NUMBERS

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Fibonacci numbers can be regarded as a formal language in a very natural way: The language $\mathcal{F}_b$ consists of words encoding each Fibonacci number in $b$-ary notation over alphabet $\{0, 1, \ldots, b-1\}$. This works for any base $b > 1$. The language $\mathcal{F}_1$ of *unary* Fibonacci numbers is defined as $\{0^{F_n} \mid n \geq 1\} = \{0, 00, 000, 00000, \ldots\}$.

The Chomsky hierarchy of formal languages consists of four levels: Regular, context-free, context-sensitive, and recursively enumerable languages (see Hopcroft and Ullman [1] for details).

It was shown by Moll and Venkatesan [2] that $\mathcal{F}_b$ is not context-free for any base $b > 1$. It is clear also that $\mathcal{F}_1$ is not context-free. If $\mathcal{F}_1$ were context-free, the Fibonacci numbers would be a semi-linear set according to Parikh's Theorem [4], i.e., a finite union of linear sets $\{nk + \ell \mid n \geq 0\}$.

Mootha [3] presented a context-sensitive grammar for $\mathcal{F}_1$, demonstrating that unary Fibonacci numbers are context-sensitive, thus placing them optimally in the Chomsky hierarchy. However, the grammar is complicated and includes 53 symbols and 80 rules. The following simpler grammar $G = (N, T, P, S)$ with nonterminals $N = \{S, A, B, B_r, C\}$, terminals $T = \{0\}$, axiom $S$, and productions

$$\begin{aligned} S &\to CS \mid B_r \\ CA &\to BC \\ CB &\to ABC \\ CB_r &\to AB_r \\ A &\to 0 \\ B &\to 0 \\ B_r &\to 0 \end{aligned}$$

also generates exactly the unary Fibonacci numbers.

It works as follows: First, a sentential form $CC \ldots CB_r$ is generated using only the rules $S \to CS$ and $S \to B_r$. Let us call a sentential form $C \ldots C\alpha B_r$ *basic*, if $\alpha$ contains only $A$'s and $B$'s. $\#_A(\beta)$ denotes the number of $A$'s and $\#_B(\beta)$ the number of $B$'s and $B_r$'s in the word $\beta$.

If $\beta$ and $\gamma$ are basic sentential forms, $\beta \stackrel{*}{\Rightarrow} \gamma$, and $\beta$ contains one more $C$ than $\gamma$, then

$$\#_A(\gamma) = \#_B(\beta) \quad \text{and} \quad \#_B(\gamma) = \#_A(\beta) + \#_B(\beta).$$

From a basic sentential form $C \ldots CB_r$ of length $n-1$, a basic sentential form of length $F_n$ will finally be derived, which itself leads to the word $0^{F_n}$. Note that the length of the derivation is

exactly $F_{n+2}+n-3$ and that the number of productions and the cardinality of $N$ and $T$ are Fibonacci numbers, too.

Observe that $b$-ary Fibonacci numbers are also context-sensitive. The context-sensitive languages are a superset of all languages that are recognizable with linear space by a deterministic Turing machine. Since addition of numbers in $b$-ary notation can be done in linear space, all Fibonacci numbers can be generated by use of the recurrence that defines Fibonacci numbers.

Finally, we note that if we change the production for nonterminal $S$ to $S \rightarrow C_\ell B_r | B_r$ and add the productions

$$C_\ell A \rightarrow C_\ell BC | BC$$
$$C_\ell B \rightarrow C_\ell ABC | ABC$$
$$C_\ell B_r \rightarrow C_\ell AB_r | AB_r$$

then the word $0^{F_n}$ can be derived in exactly $F_{n+2}-1$ steps.

## REFERENCES


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