# ON THE POSSIBILITY OF PROGRAMMING THE GENERAL 2-BY-2 MATRIX ON THE COMPLEX FIELD

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For more than thirty years, several 2-by-2 matrices have been used to discover properties of classical or generalized Fibonacci and Lucas sequences. The references [1]-[6] and their bibliographies provide a few landmarks for this area of study.

However, no one seems to have addressed the following problem: Given a linear second-order sequence $(u_n)$ defined by an arbitrary recursion on the complex field,

$$u_{n+2} - pu_{n+1} + qu_n = 0, \tag{R1}$$

and arbitrary initial values $u_0$ and $u_1$, is it possible to program the general 2-by-2 matrix on the complex field, that is, to set the entries of

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in such a way that at least one of the entries of

$$A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$$

be $u_n$ for any $n > 0$?

This could be useful to study the general sequence $(u_n)$ as it was for the classical or generalized Fibonacci and Lucas sequences recently in [5] or [6].

This note proves that the answer to this question with such a general scope is "no," and also shows that by introducing some slight restrictions, it is possible to program $A$ in such a way that both the entries $a_n$ and $d_n$ bear a close relationship to $u_n$.

***Lemma 1:*** If we set $p = a + d$ and $q = ad - bc$ in (R1), then

$$A^n = \begin{pmatrix} \frac{1}{2}L_n + \frac{1}{2}(a-d)F_n & bF_n \\ cF_n & \frac{1}{2}L_n - \frac{1}{2}(a-d)F_n \end{pmatrix}, \tag{F1}$$

$(L_n)$ and $(F_n)$ being, respectively, the Lucas and the Fibonacci sequences for (R1).

Formula (F1) is easily proved by first showing by induction that

$$A^n = \begin{pmatrix} F_{n+1} - dF_n & bF_n \\ cF_n & F_{n+1} - aF_n \end{pmatrix}$$

and then using $L_n = 2F_{n+1} - pF_n = 2F_{n+1} - (a+d)F_n$ to obtain the form (F1).

Obviously, when considered as functions of $n$, all the entries are sequences satisfying (R1). From this formula, it is clear that:

1. whatever the coefficients $p$ and $q$, we can always find $a$, $b$, $c$, and $d$ as to obtain $p = a + d$ and $q = ad - bc$. In fact, there are infinitely many solutions for $a$, $b$, $c$, and $d$;

2. we cannot obtain at will an arbitrary sequence for the entries $(b_n)$ or $(c_n)$, since they must be proportional to the Fibonacci sequence for (R1);

   In the remainder of this paper, we shall assume that $p$ and $q$ are arbitrarily chosen. As for the other entries, we have

***Lemma 2:*** If $(u_n)$ is a sequence satisfying (R1) and such that $u_0 \neq 0$, and if $a$, $b$, $c$, and $d$ satisfy the following equalities,

$$a = u_1 / u_0, \quad d = p - a, \quad bc = ad - q, \tag{E}$$

then for any $n > 0$ we have the formula:

$$A^n = \begin{pmatrix} \frac{u_n}{u_0} & bF_n \\ cF_n & L_n - \frac{u_n}{u_0} \end{pmatrix}. \tag{F2}$$

***Proof:*** The proof is necessary only for $a_n$ since, for the other entries, the result will be the consequence of (F1). Since both $(a_n)$ and $(u_n)/u_0$ satisfy (R1), it is sufficient to show that they coincide for two consecutive values of $n$. For $n = 1$, they coincide by construction. For $n = 2$, we have $a_2 = a^2 + bc = a^2 + a(p-a) - q = pa - q$; but since $a = u_1 / u_0$, we find by applying (R1) that $a_2 = u_2 / u_0$. Q.E.D.

For a given recurrence (R1) and a given sequence $(u_n)$ satisfying the conditions for the theorem, there are infinitely many corresponding matrices $A$ since $b$ and $c$ are required only to satisfy $bc = ad - q$.

We could also have programmed $d_n$ similarly by exchanging $a$ and $d$ in the above set of equalities (E).

These results may be summarized into the following theorem, which is the main aim of this paper.

***Theorem 1:***

***(a)*** Let (R1) be the general second-order linear recurrence on the complex field, $u_{n+2} - pu_{n+1} + qu_n = 0$, and $(u_n)$ any sequence satisfying (R1) and such that $u_0 \neq 0$. Then a necessary and sufficient condition for all the entries of $A^n$ considered as functions of $n > 0$ to satisfy (R1) and for the entry $(a_n)$ to be $(u_n)/u_0$ for any $n > 0$ is that

$$a = u_1 / u_0, \quad d = p - a, \quad bc = ad - q.$$

***(b)*** The other entries of $A^n$ are then determined, once $b$ and $c$ have been individually chosen in accordance with the above equalities, by

$$b_n = bF_n, \quad c_n = cF_n, \quad d_n = L_n - u_n / u_0,$$

where $L_n$ and $F_n$ are, respectively, the generalized Lucas and Fibonacci sequences of (R1).

## APPLICATIONS

As applications of (F2), we shall derive two formulas concerning the general sequence $(u_n)$ satisfying (R1).

**1.** Separation of variables for $u_{m+n}$.

By writing that for any positive integers $m$ and $n$, $A^{m+n} = A^m A^n$, and by equating the upper left entries on both sides, we obtain $u_0 u_{m+n} = u_m u_n + (u_0)^2 bcF_m F_n$.

By taking $m = n = 1$, we get $(u_0)^2 bc = u_0 u_2 - (u_1)^2$ and the formula

$$u_0 u_{m+n} = u_m u_n + \{u_0 u_2 - (u_1)^2\} F_m F_n. \tag{F3}$$

This formula has the following applications to the study of the sequence of the residues of $(u_n)$ modulo a prime when $u_n$ is an integer for any $n \geq 0$.

Let us assume that $D$ is a positive prime that divides $u_0 u_2 - (u_1)^2$. Then $D$ also divides $u_0 u_{m+n} - u_m u_n$ for any $m$ and $n \geq 0$, and therefore divides $u_0 u_{n+1} - u_1 u_n$ for any $n \geq 0$. Now define the $T$ transformation as: $T(u_n) = u_0 u_{n+1} - u_1 u_n$ for any $n$. Then, by iterating this transformation $D-1$ times, we shall prove the following theorem by exactly the same method as in [7].

***Theorem 2:*** If $D$ is a positive prime that divides $u_0 u_2 - (u_1)^2$ and is relatively prime to $u_0$, then the sequence of the residues of $(u_n)$ modulo $D$ is either constant or periodic with period $D-1$.

Thanks to the formula, $u_{n+m+1} = u_{n+1} F_{m+1} - q u_n F_m$ (which can be proved by an easy recursion), and using the method of the iterated $T$ transformation given above, we can prove

***Theorem 3:*** If $D$ is a positive prime that divides $F_m$, then the sequence of the residues of $(u_n)$ modulo $D$ is either constant or periodic with period $m(D-1)$.

This latter property is shared by any sequence of integers satisfying (R1), since we made no assumption on the values of any $u_n$, which was not the case for the former property. So there exist at least two kinds of periods for the sequences of the residues of $(u_n)$ modulo a given prime: the **universal** periods depending only on the value of the prime and which exist for any $(u_n)$, and **particular** periods depending also on the initial values of the sequence considered. For instance, if $(L_n)$ and $(F_n)$ are the classical Lucas and Fibonacci sequences, the shortest period modulo 5 for $(L_n)$ is 4, in accordance with the fact that $L_0 L_2 - (L_1)^2 = 5$, while the shortest period modulo 5 for $(F_n)$ is 20, this number also being a period [not necessarily the shortest one, as shown by $(L_n)$] for any sequence $(u_n)$, in accordance with the fact that for $m = 5$ we have $D = 5$ as divisor of $F_m$ and that $m(D-1) = 20$ in that case.

**2.** Since the $n^{\text{th}}$ power of the determinant of a square matrix is the determinant of its $n^{\text{th}}$ power, and the determinant of $A$ is $q$, we obtain, from (F2):

$$\{u_0 u_2 - (u_1)^2\}(F_n)^2 + (u_n)^2 + q^n (u_0)^2 = u_0 u_n L_n. \tag{F4}$$

This proves, for instance, that if, for any $n \geq 0$, all the sequences involved in this formula are made up of integers, and if, for a given $n$, the integer $D$ divides both $u_0$ and $F_n$, then it also divides $u_n$. Since, for any integer $m \geq 1$, $F_n$ divides $F_{mn}$, $D$ also divides $F_{mn}$ and therefore $u_{mn}$.

***Theorem 4:*** Any divisor $D$ of $u_0$ generates a sequence of zero residues of $(u_n)$ mod $D$ with a period equal to the entry point of $D$ in $(F_n)$.

[This does not mean that all the zero residues of $(u_n)$ mod $D$ are located in this sequence.]

Formula (F4) also shows that any common divisor of $u_0u_2-(u_1)^2$ and $u_n$, if also relatively prime to $u_0$, is a divisor of $q$. Therefore, if $q=\pm 1$ for any $n \geq 0$, $u_0u_2-(u_1)^2$ and $u_n$ are either prime to each other or share a common divisor with $u_0$. Then if $(L_n)$ is the generalized Lucas sequence of (R1), since we have $L_0L_2-(L_1)^2=\Delta=$ the discriminant of (R1), and $L_0=2$ always, we can state the next theorem.

***Theorem 5:*** If $q=\pm 1$ for any $n \geq 0$, then $L_n$ is relatively prime to the discriminant of its recursion (R1), provided that this discriminant is odd.

This generalizes the well-known property of the classical Lucas sequence regarding 5.

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