ON THE K^{th}-ORDER DERIVATIVE SEQUENCES OF FIBONACCI AND LUCAS POLYNOMIALS

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1. INTRODUCTION

The Fibonacci polynomials $u_n = u_n(x)$ and the Lucas polynomials $v_n = v_n(x)$ are defined by the second-order linear recurrence relations

$$u_n = xu_{n-1} + u_{n-2} \quad (u_0 = 0, u_1 = 1),$$

and

$$v_n = xv_{n-1} + v_{n-2} \quad (v_0 = 2, v_1 = x), \tag{1.1}$$

where x is an indeterminate. Their k^{th}-order derivative sequences are defined as

$$u_n^{(k)} = u_n^{(k)}(x) = \frac{d^k}{dx^k} u_n(x) \quad \text{and} \quad v_n^{(k)} = v_n^{(k)}(x) = \frac{d^k}{dx^k}(x).$$

Denote $f_n = u_n(1)$, $\ell_n = v_n(1)$, $f_n^{(k)} = u_n^{(k)}(1)$, $\ell_n^{(k)} = v_n^{(k)}(1)$. P. Filipponi and A. F. Horadam ([1], [2]) considered $f_n^{(k)}$ and $\ell_n^{(k)}$ for $k = 1, 2$ and obtained a series of results. By the end of [2], seven conjectures were presented for arbitrary k. In this paper we shall consider the more general cases, $u_n^{(k)}$ and $v_n^{(k)}$, for arbitrary k. Our results will be generalizations of the results in [1] and [2]. As special cases of our results, the seven conjectures in [2] will be proved.

Following the symbols in [1] and [2], denote $\Delta = \sqrt{x^2+4}$, $\alpha = (x+\Delta)/2$, $\beta = (x-\Delta)/2$, so that $\alpha + \beta = x$, $\alpha\beta = -1$, $\alpha - \beta = \Delta$. It is well known that

$$u_n = (\alpha^n - \beta^n)/\Delta, \quad v_n = \alpha^n + \beta^n. \tag{1.2}$$

2. EXPRESSIONS FOR $u_n^{(k)}$ AND $v_n^{(k)}$ IN TERMS OF FIBONACCI AND LUCAS POLYNOMIALS

Theorem 2.1:

$$u_n^{(k)} = \frac{k!}{2\Delta^{2k}}(a_{n,k}u_n + b_{n,k}v_n), \tag{2.1}$$

where

$$a_{n,k} = \sum_{\substack{i=0 \\ 2|k-i}}^{k} \binom{k-i+n}{k-i} \Delta^{k-i}(c_{k,i} + d_{k,i}) + \sum_{\substack{i=0 \\ 2\nmid k-i}}^{k} \binom{k-i+n}{k-i} \Delta^{k-i}(c_{k,i} - d_{k,i}), \tag{2.2}$$

and

$$b_{n,k} = \sum_{\substack{i=0 \\ 2|k-i}}^{k} \binom{k-i+n}{k-i} \Delta^{k-1-i}(c_{k,i} - d_{k,i}) + \sum_{\substack{i=0 \\ 2\nmid k-i}}^{k} \binom{k-i+n}{k-i} \Delta^{k-1-i}(c_{k,i} + d_{k,i}), \tag{2.3}$$

where $c_{k,i}$ and $d_{k,i}$ $(i = 0, 1, \ldots, k)$ satisfy the systems of linear equations

$$c_{k,i}+\binom{k+1}{1}\beta c_{k,i-1}+\cdots+\binom{k+1}{i}\beta^i c_{k,0}=(-1)^i\binom{k+1}{i}\Delta^i \tag{2.4}$$

and

$$d_{k,i}+\binom{k+1}{1}\alpha d_{k,i-1}+\cdots+\binom{k+1}{i}\alpha^i d_{k,0}=\binom{k+1}{i}\Delta^i. \tag{2.5}$$

Furthermore, for $i=0,1,\ldots,k$, there exist polynomials $p_{k,i}$ and $q_{k,i}$ in x, with integer coefficients, which satisfy

$$c_{k,i}=p_{k,i}\alpha+q_{k,i} \quad\text{and}\quad d_{k,i}=p_{k,i}\beta+q_{k,i}. \tag{2.6}$$

Proof: Let the generating functions of $\{u_n\}$ and $\{u_n^{(k)}\}$ be $U(t)=U(t,x)=\sum_{n=0}^{\infty}u_nt^n$ and $U_k(t)=U_k(t,x)=\sum_{n=0}^{\infty}u_n^{(k)}t^n$, respectively. It is well known that $U(t)=t/(1-xt-t^2)$, hence,

$$U_k(t)=\frac{\partial^k}{\partial x^k}U(t)=k!\,t^{k+1}/(1-xt-t^2)^{k+1}. \tag{2.7}$$

By partial fractions we have

$$t^{k+1}/(1-xt-t^2)^{k+1}=\sum_{i=0}^{k}Q_{k,i}/(1-\alpha t)^{k+1-i}+\sum_{i=0}^{k}R_{k,i}/(1-\beta t)^{k+1-i}, \tag{2.8}$$

where $Q_{k,i}$ and $R_{k,i}$ are independent of t. Multiplying by $\alpha^{k+1}(1-\beta t)^{k+1}$, we obtain

$$(\alpha t)^{k+1}/(1-\alpha t)^{k+1}=(\alpha+t)^{k+1}\sum_{i=0}^{k}Q_{k,i}/(1-\alpha t)^{k+1-i}+\varphi(t), \tag{2.9}$$

where the function $\varphi(t)$ is analytic at the point $t=\alpha^{-1}$ under the condition that t is considered as a complex variable (while x is a real constant). Since $(\alpha t)^{k+1}/(1-\alpha t)^{k+1}=[(1-\alpha t)^{-1}-1]^{k+1}$ and $(\alpha+t)^{k+1}=[\Delta+\beta(1-\alpha t)]^{k+1}$, we can rewrite (2.9) as

$$\sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}(1-\alpha t)^{-(k+1-i)}=\sum_{i=0}^{k+1}\binom{k+1}{i}\Delta^{k+1-i}\beta^i(1-\alpha t)^i\cdot\sum_{i=0}^{k}Q_{k,i}(1-\alpha t)^{-(k+1-i)}+\varphi(t).$$

Because of the uniqueness of the Laurent series [4] at the point $t=\alpha^{-1}$ for the function $(\alpha t)^{k+1}/(1-\alpha t)^{k+1}$, we can compare the coefficients of $(1-\alpha t)^{-(k+1-i)}$ $(i=0,1,\ldots,k)$ of the two sides in the last equality to get

$$\sum_{j=0}^{i}\binom{k+1}{j}\Delta^{k+1-j}\beta^jQ_{k,i-j}=(-1)^i\binom{k+1}{i}. \tag{2.10}$$

Let

$$Q_{k,i}=\Delta^{-(k+1+i)}c_{k,i}\quad(i=0,1,\ldots,k) \tag{2.11}$$

and substitute it into (2.10); then we get (2.4). For the same reason, it follows that

$$\sum_{j=0}^{i}\binom{k+1}{j}(-\Delta)^{k+1-j}\alpha^jR_{k,i-j}=(-1)^i\binom{k+1}{i}. \tag{2.12}$$

Let

$$R_{k,i}=(-\Delta)^{-(k+1+i)}d_{k,i}\quad(i=0,1,\ldots,k) \tag{2.13}$$

and substitute it into (2.12); then we get (2.5).

Now we shall prove (2.6). From (2.4) and (2.5), $c_{k,0}=d_{k,0}=1$; hence, the conclusion holds for $i=0$. Suppose the conclusion holds for $0,1,\dots,i-1$. Then, from (2.4) and (2.5), we have

$$c_{k,i}=(-1)^i\binom{k+1}{i}\Delta^i-\sum_{j=1}^{i}\binom{k+1}{j}\beta^j(p_{k,i-j}\alpha+q_{k,i-j}) \tag{2.14}$$

and

$$d_{k,i}=\binom{k+1}{i}\Delta^i-\sum_{j=1}^{i}\binom{k+1}{j}\alpha^j(p_{k,i-j}\beta+q_{k,i-j}). \tag{2.15}$$

From (1.2), it is easy to show that $\beta^j=-u_j\alpha+u_{j+1}$; hence,

$$\begin{aligned}\beta^j(p_{k,i-j}\alpha+q_{k,i-j})&=-p_{k,i-j}\beta^{j-1}+q_{k,i-j}\beta^j\\&=(p_{k,i-j}u_{j-1}-q_{k,i-j}u_j)\alpha+(q_{k,i-j}u_{j+1}-p_{k,i-j}u_j).\end{aligned}$$

For the same reason, we have

$$\alpha^j(p_{k,i-j}\beta+q_{k,i-j})=(p_{k,i-j}u_{j-1}-q_{k,i-j}u_j)\beta+(q_{k,i-j}u_{j+1}-p_{k,i-j}u_j).$$

We can see that Δ^i is a polynomial in x with integer coefficients for $2\,|\,i$, but $\Delta^i=\Delta^{i-1}(x-2\beta)$ and $(-\Delta)^i=\Delta^{i-1}(x-2\alpha)$ for $2\nmid i$. By substituting the above results into (2.14) and (2.15), and by the inductive hypothesis, the conclusion is proved.

Now substituting (2.11), (2.13), and (2.6) into (2.8), then into (2.7), we get

$$\begin{aligned}U_k(t)&=\frac{k!}{\Delta^{2k}}\left[\sum_{i=0}^{k}c_{k,i}\Delta^{k-1-i}/(1-\alpha t)^{k+1-i}+\sum_{i=0}^{k}d_{k,i}(-\Delta)^{k-1-i}/(1-\beta t)^{k+1-i}\right]\\&=\frac{k!}{\Delta^{2k}}\left[\sum_{2|k-i}(c_{k,i}/(1-\alpha t)^{k+1-i}-d_{k,i}/(1-\beta t)^{k+1-i})\Delta^{k-1-i}\right.\\&\qquad\left.+\sum_{2\nmid k-i}(c_{k,i}/(1-\alpha t)^{k+1-i}+d_{k,i}/(1-\beta t)^{k+1-i})\Delta^{k-1-i}\right].\end{aligned}$$

Expanding the right side of the last expression into power series in t and using (2.6), we obtain

$$u_n^{(k)}=\frac{k!}{\Delta^{2k}}\left[\sum_{2|k-i}\binom{k-i+n}{k-i}\Delta^{k-i}(p_{k,i}u_{n+1}+q_{k,i}u_n)+\sum_{2\nmid k-i}\binom{k-i+n}{k-i}\Delta^{k-1-i}(p_{k,i}v_{n+1}+q_{k,i}v_n)\right]. \tag{2.16}$$

It is easy to prove that $u_{n+1}=(xu_n+v_n)/2$, $v_{n+1}=(\Delta^2u_n+xv_n)/2$; hence,

$$\begin{aligned}p_{k,i}u_{n+1}+q_{k,i}u_n&=((p_{k,i}x+2q_{k,i})u_n+p_{k,i}v_n)/2\\&=((c_{k,i}+d_{k,i})u_n+(c_{k,i}-d_{k,i})\Delta^{-1}v_n)/2,\end{aligned} \tag{2.17}$$

$$\begin{aligned}p_{k,i}v_{n+1}+q_{k,i}v_n&=(p_{k,i}\Delta^2u_n+(p_{k,i}x+2q_{k,i})v_n)/2\\&=((c_{k,i}-d_{k,i})\Delta u_n+(c_{k,i}+d_{k,i})v_n)/2.\end{aligned} \tag{2.18}$$

Substitute (2.17) and (2.18) into (2.16) and we are done. □

As an example, when $k = 3$ and 4, Theorem 2.1 gives the following results:

$$c_{30} = d_{30} = 1, \qquad c_{31} = -4\Delta - 4\beta,\ d_{31} = 4\Delta - 4\alpha,$$
$$c_{32} = 6\Delta^2 + 16\beta\Delta + 10\beta^2, \qquad d_{32} = 6\Delta^2 - 16\alpha\Delta + 10\alpha^2,$$
$$c_{33} = -4\Delta^3 - 24\beta\Delta^2 - 40\beta^2\Delta - 20\beta^3, \qquad d_{33} = 4\Delta^3 - 24\alpha\Delta^2 + 40\alpha^2\Delta - 20\alpha^3,$$
$$c_{30} + d_{30} = 2, \qquad c_{31} + d_{31} = -4x,$$
$$c_{32} + d_{32} = 6x^2 + 4, \qquad c_{33} + d_{33} = -4x^3 + 4x,$$
$$c_{30} - d_{30} = 0, \qquad c_{31} - d_{31} = -4\Delta,$$
$$c_{32} - d_{32} = 6x\Delta, \qquad c_{33} - d_{33} = (-4x^2 + 4)\Delta,$$

$$\begin{aligned} a_{n3} &= \binom{2+n}{2}\Delta^2(-4x) + \binom{0+n}{0}(-4x^3 + 4x) + \binom{3+n}{3}\Delta^3 \cdot 0 + \binom{1+n}{1}\Delta \cdot 6x\Delta \\ &= -2(n^2+1)x^3 - 4(2n^2 - 3)x, \end{aligned}$$

$$\begin{aligned} b_{n3} &= \binom{2+n}{2}\Delta(-4\Delta) + \binom{0+n}{0}\Delta^{-1}(-4x^3 + 4)\Delta + \binom{3+n}{3}\Delta^2 \cdot 2 + \binom{1+n}{1}(6x^2 + 4) \\ &= \frac{1}{3}n(n^2+11)x^2 + \frac{4}{3}n(n^2 - 4), \end{aligned}$$

$$u_n^{(3)} = [-(6(n^2+1)x^3 + 12(2n^2 - 3)x)u_n + (n(n^2+11)x^2 + 4n(n^2 - 4))v_n] / \Delta^6; \tag{2.19}$$

in particular,

$$f_n^{(3)} = (n^2 - 1)(n\ell_n - 6f_n)/25. \tag{2.20}$$

$$c_{40} = d_{40} = 1, \qquad c_{41} = -5\Delta - 5\beta,\ d_{41} = 5\Delta - 5\alpha,$$
$$c_{42} = 10\Delta^2 + 25\beta\Delta + 15\beta^2, \qquad d_{42} = 10\Delta^2 - 25\alpha\Delta + 15\alpha^2,$$
$$c_{43} = -10\Delta^3 - 50\beta\Delta^2 - 75\beta^2\Delta - 35\beta^3, \qquad d_{43} = 10\Delta^3 - 50\alpha\Delta^2 + 75\alpha^2\Delta - 35\alpha^3,$$
$$c_{44} = 5\Delta^4 + 50\beta\Delta^3 + 150\beta^2\Delta^2 + 175\beta^3\Delta + 70\beta^4,$$
$$d_{44} = 5\Delta^4 - 50\alpha\Delta^3 + 150\alpha^2\Delta^2 - 175\alpha^3\Delta + 70\alpha^4,$$
$$c_{40} + d_{40} = 2, \qquad c_{41} + d_{41} = -5x,$$
$$c_{42} + d_{42} = 10x^2 + 10, \qquad c_{43} + d_{43} = -10x^3 - 5x,$$
$$c_{44} + d_{44} = 5x^4 - 15x^2, \qquad c_{40} - d_{40} = 0,$$
$$c_{41} - d_{41} = -5\Delta, \qquad c_{42} - d_{42} = 10x\Delta,$$
$$c_{43} - d_{43} = (-10x^2 + 5)\Delta, \qquad c_{44} - d_{44} = (5x^3 - 15x)\Delta,$$

$$\begin{aligned} a_{n4} &= \binom{4+n}{4}\Delta^4 \cdot 2 + \binom{2+n}{2}\Delta^2(10x^2 + 10) + \binom{0+n}{0}(5x^4 - 15x^2) \\ &\quad + \binom{3+n}{3}\Delta^3(-5\Delta) + \binom{1+n}{1}\Delta(-10x^2 + 5)\Delta \\ &= \frac{1}{12}(n^4 + 35n^2 + 24)x^4 + \frac{1}{3}(2n^4 + 25n^2 - 72)x^2 + \frac{4}{3}(n^4 - 10n^2 + 9), \end{aligned}$$

$$b_{n4} = \binom{4+n}{4}\Delta^3\cdot 0 + \binom{2+n}{2}\Delta(10x\Delta) + \binom{0+n}{0}\Delta^{-1}(5x^3-15x)\Delta$$
$$+\binom{3+n}{3}\Delta^2(-5x) + \binom{1+n}{1}(-10x^3-5x)$$
$$= -\frac{5}{6}n(n^2+5)x^3 - \frac{5}{3}n(2n^2-11)x,$$

$$u_n^{(4)} = [((n^4+35n^2+24)x^4 + 4(2n^4+25n^2-72)x^2 + 16(n^4-10n^2+9))u_n - (10n(n^2+5)x^3 + 20n(2n^2-11)x)v_n]/\Delta^8; \tag{2.21}$$

in particular,

$$f_n^{(4)} = [(5n^4-5n^2-24)f_n - 2n(5n^2-17)\ell_n]/125. \tag{2.22}$$

We observe that (2.6) can be verified by using the above results.

From $v_n^{(k)} = nu_n^{(k-1)}$ (see 1° of Theorem 3.1 in the next section) and Theorem 2.1, we can obtain the expression for $v_n^{(k)}$ in terms of u_n and v_n.

3. SOME IDENTITIES INVOLVING $u_n^{(k)}$ AND $v_n^{(k)}$

If we differentiate certain identities involving u_n and v_n, we can get the corresponding identities involving $u_n^{(k)}$ and $v_n^{(k)}$.

Theorem 3.1:

1°. $v_n^{(k)} = nu_n^{(k-1)}$; (3.1)

2°. $u_n^{(k)} = xu_{n-1}^{(k)} + u_{n-2}^{(k)} + ku_{n-1}^{(k-1)}$, $v_n^{(k)} = xv_{n-1}^{(k)} + v_{n-2}^{(k)} + kv_{n-1}^{(k-1)}$; (3.2)

3°. $v_n^{(k)} = u_{n+1}^{(k)} + u_{n-1}^{(k)}$, (3.3)

$$\Delta^2 u_n^{(k)} + 2kxu_n^{(k-1)} + k(k-1)u_n^{(k-2)} = v_{n+1}^{(k)} + v_{n-1}^{(k)}; \tag{3.4}$$

4°.
$$u_{m+n}^{(k)} = \sum_{i=0}^{k}\binom{k}{i}(u_{m+1}^{(k-i)}u_n^{(i)} + u_m^{(k-i)}u_{n-1}^{(i)}), \tag{3.5}$$

$$v_{m+n}^{(k)} = \sum_{i=0}^{k}\binom{k}{i}(v_{m+1}^{(k-i)}u_n^{(i)} + v_m^{(k-i)}u_{n-1}^{(i)}), \tag{3.6}$$

$$u_{m-n}^{(k)} = (-1)^n\sum_{i=0}^{k}\binom{k}{i}(u_m^{(k-i)}u_{n+1}^{(i)} - u_{m+1}^{(k-i)}u_n^{(i)}), \tag{3.7}$$

$$v_{m-n}^{(k)} = (-1)^n\sum_{i=0}^{k}\binom{k}{i}(v_{m+1}^{(k-i)}u_n^{(i)} - u_m^{(k-i)}v_{n+1}^{(i)}); \tag{3.8}$$

in particular,

$$u_{-n}^{(k)} = (-1)^{n-1}u_n^{(k)}; \tag{3.9}$$

$$v_{-n}^{(k)} = (-1)^n v_n^{(k)}; \tag{3.10}$$

$$u_{2n}^{(k)} = \sum_{i=0}^{k}\binom{k}{i}u_n^{(k-i)}v_n^{(i)}; \tag{3.11}$$

$$v_{2n}^{(k)} = 2\sum_{i=0}^{k-1}\binom{k-1}{i}v_n^{(k-i)}v_n^{(i)}; \tag{3.12}$$

$$u_{2n+1}^{(k)} = \sum_{i=0}^{k}\binom{k}{i}u_{n+1}^{(k-i)}v_n^{(i)}; \tag{3.13}$$

$$v_{2n+1}^{(k)} = \sum_{i=0}^{k}\binom{k}{i}v_{n+1}^{(k-i)}v_n^{(i)} - (-1)^n\delta_{k,1} \text{ (δ is the Kronecker function)}; \tag{3.14}$$

5°. $$u_{m+n}^{(k)} + (-1)^n u_{m-n}^{(k)} = \sum_{i=0}^{k}\binom{k}{i}u_m^{(k-i)}v_n^{(i)}; \tag{3.15}$$

$$u_{m+n}^{(k)} - (-1)^n u_{m-n}^{(k)} = \sum_{i=0}^{k}\binom{k}{i}v_m^{(k-i)}u_n^{(i)}; \tag{3.16}$$

$$v_{m+n}^{(k)} + (-1)^n v_{m-n}^{(k)} = \sum_{i=0}^{k}\binom{k}{i}v_m^{(k-i)}v_n^{(i)}; \tag{3.17}$$

$$v_{m+n}^{(k)} - (-1)^n v_{m-n}^{(k)} = \sum_{i=0}^{k}\binom{k}{i}u_m^{(k-i)}(v_{n+1}^{(i)} + v_{n-1}^{(i)}); \tag{3.18}$$

6°. $$xv_n^{(k)} = (n-k+1)v_n^{(k-1)} - 2(v_{n-1}^{(k)} + u_{n-1}^{(k-1)}). \tag{3.19}$$

Proof: **1°**. This can be obtained by differentiating the identity $v_n^{(1)} = nu_n$, which had been proved in [1].

2°. By differentiating (1.1).

3° ~ 5°. By differentiating the following identities, which can be seen in [5] or can be derived from (1.2):

$$\begin{array}{ll}
v_n = u_{n+1} + u_{n-1}, & \Delta^2 u_n = v_{n+1} + v_{n-1}, \\
u_{m+n} = u_{m+1}u_n + u_m u_{n-1}, & v_{m+n} = v_{m+1}u_n + v_m u_{n-1}, \\
u_{m-n} = (-1)^n(u_m u_{n+1} - u_{m+1}u_n), & v_{m-n} = (-1)^n(u_{m+1}v_n - u_m v_{n+1}), \\
u_{m+n} + (-1)^n u_{m-n} = u_m v_n, & u_{m+n} - (-1)^n u_{m-n} = v_m u_n, \\
v_{m+n} + (-1)^n v_{m-n} = v_m v_n, & v_{m+n} - (-1)^n v_{m-n} = \Delta^2 u_m u_n = u_m(v_{n+1} + v_{n-1}), \\
u_{-n} = (-1)^{n-1}u_n, & v_{-n} = (-1)^n v_n, \\
u_{2n} = u_n v_n, & v_{2n} = v_n^2 - 2(-1)^n, \\
u_{2n+1} = u_{n+1}v_n - (-1)^n, & v_{2n+1} = v_{n+1}v_n - (-1)^n x.
\end{array}$$

6°. From the well-known identity $v_n = xu_n + 2u_{n-1}$, we get $xnu_n = nv_n - 2((n-1)u_{n-1} + u_{n-1})$, that is, $xv_n^{(1)} = nv_n - 2(v_{n-1}^{(1)} + u_{n-1})$, and the proof is finished by differentiating the last expres sion. □

Let $x = 1$ in 1°, 2°, 3°, and 6° of Theorem 3.1; then Conjectures 1-5 in [2] and [3] are proved.

4. SOME CONGRUENCE RELATIONS AND MODULAR PERIODICITIES

First, we introduce some concepts and lemmas. Set polynomials

$$g(t) = t^k - a_1 t^{k-1} - \cdots - a_{k-1}t - a_k \tag{4.1}$$

and

$$\tilde{g}(t) = 1 - a_1 t - \cdots - a_{k-1}t^{k-1} - a_k t^k. \tag{4.2}$$

Obviously, $g(t)=t^k\tilde{g}(1/t)$ and $\tilde{g}(t)=t^k g(1/t)$. The set of homogeneous linear recurrence sequences $\{g_n\}$ of order k [each of which has $g(t)$ as its characteristic polynomial] defined by

$$g_{n+k}=a_1g_{n+k-1}+\cdots+a_{k-1}g_{n+1}+a_kg_n \tag{4.3}$$

is denoted by $\Omega(g(t))=\Omega(a_1,\dots,a_k)$. The sequence $\{w_n\}\in\Omega(g(t))$ is called **the principal sequence** in $\Omega(g(t))$ if it has the initial values $w_0=w_1=\cdots=w_{k-2}=0,\ w_{k-1}=1$.

Lemma 4.1: Let $\{w_n\}$ be the principal sequence in $\Omega(g(t))$; then its generating function is

$$W(t)=t^{k-1}/\tilde{g}(t) \tag{4.4}$$

(see [6], p. 137).

In the following discussions, we suppose that $a_1,\dots,a_k$ are all integers. Let $\{g_n\}$ be an integer sequence in $\Omega(g(t))$ and m be an integer greater than one. Denote the period of $\{g_n\}$ modulo m by $P(m, g_n)$. If there exists a positive integer λ such that

$$t^\lambda\equiv 1\ (\text{mod}\, m,\, g(t)), \tag{4.5}$$

then the least positive integer λ such that (4.5) holds is called **the period of $g(t)$ modulo m** and is denoted by $P(m, g(t))$.

We point out that

$$P(m, g(t))=P(m,\tilde{g}(t)) \text{ for } \gcd(m, a_k)=1. \tag{4.6}$$

To show (4.6), it is sufficient to show that $g(t)|(t^\lambda-1)$ (mod m) iff $\tilde{g}(t)|(t^\lambda-1)$ (mod m). Assume that $\tilde{g}(t)|(t^\lambda-1)$ (mod m). Then we have $t^\lambda-1=h(t)\tilde{g}(t)+m\cdot r(t)$, where $h(t)$ and $r(t)\in Z(t)$ (the set of polynomials with integer coefficients). Replacing t with $1/t$, we obtain $(1/t)^\lambda-1=h(1/t)\tilde{g}(1/t)+m\cdot r(1/t)$. Multiplying by t^λ, we then have $-(t^\lambda-1)=t^{\lambda-k}h(1/t)g(t)+m\cdot t^\lambda r(1/t)$. Since $\gcd(m, a_k)=1$, the degree of $\tilde{g}(t)$ (mod m) is k. This leads to $t^{\lambda-k}h(1/t)$ and $t^\lambda r(1/t)\in Z(t)$. Hence, $g(t)|(t^\lambda-1)$ (mod m). The converse can be proved in the same way.

Let $B(t)=1/\tilde{g}(t)=\sum_{n=0}^\infty b_nt^n$. Let $\{w_n\}$ be the principal sequence in $\Omega(g(t))$. Then, from (4.4), we have $w_n=b_{n-k+1}$; and therefore, $P(m, w_n)=P(m, b_n)$. Corollary 2 in [7] means that $P(m, b_n)=P(m,\tilde{g}(t))$.* Therefore,

$$P(m, w_n)=P(m,\tilde{g}(t)). \tag{4.7}$$

From (4.6) and (4.7), we obtain

Lemma 4.2: Let $\{w_n\}$ be the principal sequence in $\Omega(g(t))=\Omega(a_1,\dots,a_k)$, $\gcd(m, a_k)=1$. Then

$$P(m, w_n)=P(m, g(t)). \tag{4.8}$$

Using the footnote and (4.6), Theorems 17, 21, and 15 in [7] can be rewritten as Lemmas 4.3, 4.4, and 4.5, respectively.

* In [7] the period of $\{b_n\}$ modulo m is referred to as the period of its generating function $B(t)=1/\tilde{g}(t)$ modulo m. Hence, the concept "the period of $1/\tilde{g}(t)$ modulo m" stated in [7] should be translated into "$P(m,\tilde{g}(t))$" in this paper.

Lemma 4.3: Let $\varphi(t)$ be a monic polynomial with integer coefficients, p be a prime, $p \nmid \varphi(0)$, and $\varphi(t)$ be irreducible modulo p; then, for $p^{r-1} < s \le p^r$ $(r \ge 1)$,

$$P(p^m, \varphi(t)^s) = p^{m+r-1} \cdot P(p, \varphi(t)). \tag{4.9}$$

Lemma 4.4: Let $\varphi(t)$ be a monic polynomial with integer coefficients, p be an odd prime, $p \nmid \varphi(0)$, and $\varphi(t)$ be irreducible modulo p. Assume $h_\tau(t) = \prod_{i=1}^{\tau} \Psi_i(t)$, where $\Psi_i(t) \equiv \varphi(t)^s$ (mod p) $(i = 1, \dots, \tau)$. For fixed $s, r \ge 1$, if there exists an integer $T > 1$ such that

$$(T-1)s \le p^{r-1} < Ts < (T+1)s \le p^r, \tag{4.10}$$

then, for every τ satisfying $p^{r-1} < \tau s \le p^r$, it follows that

$$P(p^m, h_\tau(t)) = P(p^m, \varphi(t)^{\tau s}) = p^{m+r-1} \cdot P(p, \varphi(t)). \tag{4.11}$$

Lemma 4.5: Let $\varphi(t)$ be a monic polynomial with integer coefficients, p be an odd prime, $p \nmid \varphi(0)$. If $P(p, \varphi(t)) = P(p^2, \varphi(t)) = \cdots = P(p^i, \varphi(t)) \ne P(p^{i+1}, \varphi(t))$, then $m > i$ leads to

$$P(p^m, \varphi(t)) = p^{m-i} \cdot P(p^i, \varphi(t)). \tag{4.12}$$

Lemma 4.6: Let p be an odd prime, for $j = 1, 2$, $\varphi_j(t)$ be a monic polynomial with integer coefficients, $p \nmid \varphi_j(0)$, and $\varphi_j(t)$ be irreducible modulo p. Assume $h_\tau(t) = \prod_{i=1}^{\tau} \Psi_i(t)$, where $\Psi_i(t) \equiv \varphi_1(t)^s \varphi_2(t)^s$ (mod p) $(i = 1, \dots, \tau)$, $\gcd(\varphi_1(t), \varphi_2(t)) = 1$ (mod p). For fixed $s, r \ge 1$, if there exists an integer $T > 1$ such that (4.10) holds, then for every τ satisfying $p^{r-1} < \tau s \le p^r$ it follows that

$$P(p^m, h_\tau(t)) = P(p^m, \varphi_1(t)^{\tau s} \varphi_2(t)^{\tau s}) = p^{m+r-1} \cdot \operatorname{lcm}\{P(p, \varphi_1(t)), P(p, \varphi_2(t))\}. \tag{4.13}$$

Proof: Denote $P(p, \varphi_j(t)) = \lambda_j$ $(j = 1, 2)$, $\operatorname{lcm}\{\lambda_1, \lambda_2\} = \lambda$. Since $h_\tau(t) \equiv \varphi_1(t)^{\tau s} \varphi_2(t)^{\tau s}$ (mod p), $\gcd(\varphi_1(t), \varphi_2(t)) = 1$ (mod p), we have $P(p, h_\tau(t)) = \operatorname{lcm}\{P(p, \varphi_1(t)^{\tau s}), P(p, \varphi_2(t)^{\tau s})\}$. By Lemma 4.3, $P(p, \varphi_j(t)^{\tau s}) = p^r \lambda_j$; hence, $P(p, h_\tau(t)) = p^r \lambda$.

Because T is the least τ satisfying $p^{r-1} < \tau s \le p^r$ from (4.10), we get $h_T(t) | h_\tau(t)$; therefore, $P(p^m, h_T(t)) | P(p^m, h_\tau(t))$. By Lemma 4.5, $P(p^m, h_\tau(t)) | p^{m-1} \cdot P(p, h_\tau(t)) = p^{m+r-1}\lambda$. By the same lemma, if we can show $P(p^2, h_T(t)) \ne P(p, h_T(t)) = p^r \lambda$, then $P(p^m, h_T(t)) = p^{m+r-1}\lambda$ and (4.13) holds.

Now we can rewrite $\Psi_i(t) = \varphi_1(t)^s \varphi_2(t)^s - p\theta_i(t)$, $i = 1, \dots, T$. Hence,

$$h_T(t) \equiv \varphi_1(t)^{sT} \varphi_2(t)^{sT} - p\varphi_1(t)^{s(T-1)} \cdot \varphi_2(t)^{s(T-1)} \cdot \zeta(t) \pmod{p^2}, \text{ where } \zeta(t) = \sum_{i=0}^{T} \theta_i(t).$$

Then $h_T(t)[\varphi_1(t)^s \varphi_2(t)^s + p\zeta(t)] \equiv \varphi_1(t)^{sT+s} \varphi_2(t)^{sT+s}$ (mod p^2). Therefore,

$$\frac{t^{p^r\lambda} - 1}{h_T(t)} \equiv \frac{t^{p^r\lambda} - 1}{\varphi_1(t)^{sT} \varphi_2(t)^{sT}} + \frac{p(t^{p^r\lambda} - 1)\zeta(t)}{\varphi_1(t)^{sT+s} \varphi_2(t)^{sT+s}} \pmod{p^2}. \tag{4.14}$$

From (4.10) and Lemma 4.3, we know that $P(p, \varphi_j(t)^{sT+s}) = p^r \cdot P(p, \varphi_j(t)) = p^r \lambda_j$; thus, $\varphi_j(t)^{sT+s} | (t^{p^r\lambda} - 1)$ (mod p). From $\gcd(\varphi_1(t), \varphi_2(t)) = 1$ (mod p), it follows that

$$\varphi_1(t)^{sT+s} \varphi_2(t)^{sT+s} | (t^{p^r\lambda} - 1) \pmod{p},$$

and so

$$\varphi_1(t)^{sT+s}\varphi_2(t)^{sT+s} \mid p(t^{p^r\lambda}-1) \pmod{p^2}.$$

Assume that $P(p^2, h_T(t)) = p^r\lambda$, then $h_T(t) \mid (t^{p^r\lambda}-1) \pmod{p^2}$. From equation (4.14), we get $\varphi_j(t)^{sT} \mid (t^{p^r\lambda}-1) \pmod{p^2}$; this leads to $P(p^2, \varphi_j(t)^{sT}) \mid p^r\lambda$. But from Lemma 4.3 we have $P(p^2, \varphi_j(t)^{sT}) = p^{r+1}\lambda_j$. This leads to the contradiction that $p^{r+1}\lambda \mid p^r\lambda$. □

In the following discussions of this section when the divisibilities of $u_n^{(k)}$ and $v_n^{(k)}$ are considered, we assume x takes integer values only.

Theorem 4.1:

$$u_n^{(k)} \equiv v_n^{(k)} \equiv 0 \pmod{k!}. \tag{4.15}$$

Proof: Denote

$$F_k(t) = (t^2 - xt - 1)^{k+1}. \tag{4.16}$$

Let $\{w_n\}$ be the principal sequence in $\Omega(F_k(t))$. From Lemma 4.1, the generating function of $\{w_n\}$ is

$$W(t) = t^{2k+1} / (1 - xt - t^2)^{k+1}. \tag{4.17}$$

Comparing (2.7) to (4.17), we get

$$u_n^{(k)} = k!\, w_{n+k}. \tag{4.18}$$

Because $\{w_n\}$ is an integer sequence, we have $u_n^{(k)} \equiv 0 \pmod{k!}$, and from (3.3) we get $v_n^{(k)} \equiv 0 \pmod{k!}$. □

Theorem 4.2:

$$v_n^{(k)} \equiv 0 \pmod{n} \ (k \ge 1). \tag{4.19}$$

This follows from (3.1).

The results of the last two theorems are generalizations of the results of Conjectures 6-7 in [2].

Theorem 4.3: Let p be an odd prime, $p > k$.

1°. If $p \nmid \Delta^2$, then

$$P(p^m, u_n^{(k)}) = P(p^m, v_n^{(k)}) = p^m \cdot P(p, u_n) = p^m \cdot P(p, v_n). \tag{4.20}$$

2°. If $p \mid \Delta^2$ and $p^{r-1} < 2k+2 < p^r$ $(r = 1 \text{ or } 2)$, then

$$P(p^m, u_n^{(k)}) = 4p^{m+r-1}. \tag{4.21}$$

3°. If $p \mid \Delta^2$ and $p^{r-1} < 2k < p^r$ $(r = 1 \text{ or } 2)$, then

$$P(p^m, v_n^{(k)}) = 4p^{m+r-1}. \tag{4.22}$$

Proof: Denote $f(t) = t^2 - xt - 1$. From Lemma 4.2, (4.18), and (4.16), for $p > k$, we have $P(p, u_n) = P(p, f(t))$ and $P(p^m, u_n^{(k)}) = P(p^m, F_k(t))$.

1°. Let $p \nmid \Delta^2$. From $v_n = u_{n+1} + u_{n-1}$ and $\Delta^2 u_n = v_{n+1} + v_{n-1}$, it follows that $P(p, u_n) = P(p, v_n)$ $= \lambda$.

When $f(t)$ is irreducible modulo p, the conclusion $P(p^m, u_n^{(k)}) = p^m\lambda$ can be proved by letting $\varphi(t) = f(t), s = k+1, r = 1$ in Lemma 4.3. When $f(t) \equiv (t-a)(t-b), a \not\equiv b \pmod p$, the same conclusion can be proved by letting $\varphi_1(t) = t-a, \varphi_2(t) = t-b, s = r = 1, \tau = k+1$ in Lemma 4.6.

We now prove $P(p^m, v_n^{(k)}) = p^m\lambda$. From (3.3), we can see that $P(p^m, v_n^{(k)}) | P(p^m, u_n^{(k)})$. On the other hand, from $u_n = (v_{n+1} + v_{n-1}) / \Delta^2$, by differentiating, we can obtain

$$u_n^{(k)} = \sum_{i=0}^{k} \binom{k}{i} (v_{n+1}^{(k-i)} + v_{n-1}^{(k-i)}) M_i(x) / \Delta^{2i+2}, \tag{4.23}$$

where $M_i(x)$ is a polynomial in x with integer coefficients that are independent of n. We see that (3.2) implies $P(p^m, v_n^{(i-1)}) | P(p^m, v_n^{(i)})$. Hence, for $i = 0, 1, \ldots, k$, $P(p^m, v_n^{(k-i)}) | P(p^m, v_n^{(k)})$. From (4.23), it follows that $P(p^m, u_n^{(k)}) | P(p^m, v_n^{(k)})$. Thus, $P(p^m, v_n^{(k)}) = P(p^m, u_n^{(k)}) = p^m\lambda$.

2°. Let $p | \Delta^2$, then $f(t) \equiv (t - x/2)^2 \pmod p$. From $x^2 \equiv -4$, we get $(x/2)^2 \equiv -1 \pmod p$. Hence, $P(p, t - x/2) = \text{ord}_p(x/2) = 4$.* In Lemma 4.4, if we take $\varphi(t) = t - x/2$, $h_\tau(t) = F_k(t) \equiv \varphi(t)^{2k+2} \pmod p$, $s = 2, r = 1$ or $2, \tau = k+1$, then we get the required result.

3°. Using the result of 2°, it follows that $P(p^m, v_n^{(k)}) = P(p^m, nu_n^{(k-1)}) | \text{lcm}\{P(p^m, n), P(p^m, u_n^{(k-1)})\} = 4p^{m+r-1}$ when $p^{r-1} < 2k < p^r$ ($r = 1$ or 2). Since $v_n = \alpha^n + \beta^n \equiv 2(x/2)^n \pmod p$, then $4 = P(p, v_n) | P(p^m, v_n^{(k)})$, and we have $P(p^m, v_n^{(k)}) = 4p^M$. We want to show that $M = m + r - 1 = m+1$ for $r = 2$, or $= m$ for $r = 1$. First, let $r = 2$. If it would not be the case, that is, if $M \le m$, then if we replace n by $n + 4p^m$ in (3.19) we have

$$xv_n^{(k)} \equiv (n + 4p^m - k + 1)v_n^{(k-1)} - 2[v_{n-1}^{(k)} + u_{n+4p^m-1}^{(k-1)}] \pmod{p^m}.$$

Subtracting this from $xv_n^{(k)} \equiv (n-k+1)v_n^{(k-1)} - 2[v_{n-1}^{(k)} + u_{n-1}^{(k-1)}] \pmod{p^m}$, we get $u_{n+4p^m-1}^{(k-1)} - u_{n-1}^{(k-1)} \equiv 2p^m v_n^{(k-1)} \equiv 0 \pmod{p^m}$. This means that $P(p^m, u_n^{(k-1)}) | 4p^m$ for $r = 2$. But, by 2°, we should have $P(p^m, u_n^{(k-1)}) = 4p^{m+1}$ for $r = 2$. A contradiction!

Next, let $r = 1$. The least k satisfying $1 < 2k < p$ is 1. Recalling that $P(p^m, v_n^{(1)}) | P(p^m, v_n^{(k)})$, we need only prove that $M = m$ for $k = 1$. On the contrary, suppose $M \le m - 1$. then

$$v_{n+4p^{m-1}}^{(1)} - v_n^{(1)} = (n + 4p^{m-1})u_{n+4p^{m-1}} - nu_n \equiv 0 \pmod{p^m}.$$

Expanding u_n in (1.2) into the polynomial in x, Δ, and noting $p | \Delta^2$, we obtain

$$nu_n = n\sum_{i=0}^{[(n-1)/2]} \binom{n}{2i+1} (x/2)^{n-2i-1} (\Delta/2)^{2i} \equiv n\sum_{i=0}^{m-1} \binom{n}{2i+1} (x/2)^{n-2i-1} (\Delta/2)^{2i} \pmod{p^m} \tag{4.24}$$

and

$$(n + 4p^{m-1})u_{n+4p^{m-1}} \equiv (n + 4p^{m-1}) \sum_{i=0}^{m-1} \binom{n + 4p^{m-1}}{2i+1} (x/2)^{n+4p^{m-1}-2i-1} (\Delta/2)^{2i} \pmod{p^m}. \tag{4.25}$$

When $m > 1$, since

---

* Let m and a be integers greater than one, $\gcd(m, a) = 1$. The least positive integer λ satisfying $a^\lambda \equiv 1 \pmod m$ is called the order of a modulo m and is denoted by $\text{ord}_m(a)$. Since $t^\lambda - 1 = [(t-a) + a]^\lambda - 1 \equiv a^\lambda - 1 \pmod{(t-a)}$, we have $P(m, t-a) = \text{ord}_m(a)$.

$$(n+4p^{m-1})\binom{n+4p^{m-1}}{2i+1} \equiv n\binom{n}{2i+1} \pmod{p^{m-1}} \quad \text{and} \quad p \mid \Delta^{2i} \text{ for } i \geq 1,$$

and furthermore, $(x/2)^4 = 1 \pmod{p}$ implies $(x/2)^{4p^{m-1}} \equiv 1 \pmod{p^m}$, (4.25) can be reduced to

$$(n+4p^{m-1})u_{n+4p^{m-1}} \equiv (n+4p^{m-1})^2(x/2)^{n-1} + n\sum_{i=1}^{m-1}\binom{n}{2i+1}(x/2)^{n-2i-1}(\Delta/2)^{2i} \pmod{p^m}. \quad (4.26)$$

Subtract (4.24) from (4.26) to get

$$(n+4p^{m-1})u_{n+4p^{m-1}} - nu_n \equiv 8np^{m-1}(x/2)^{n-1} \not\equiv 0 \pmod{p^m} \ \text{ for } p \nmid n.$$

This is a contradiction!

When $m = 1$, from (4.24) and (4.25), we obtain

$$(n+4)u_{n+4} - nu_n \equiv 8(n+2)(x/2)^{n-1} \not\equiv 0 \pmod{p}$$

for $n \not\equiv -2 \pmod{p}$. This is also a contradiction! □

From Theorem 4.3, we can obtain many specific congruences. For this, we introduce another concept. Let $\{g_n\}$ be an integer sequence. If there exists a positive integer s, a nonnegative integer n_0, and an integer c, $\gcd(m, c) = 1$, such that

$$g_{n+s} \equiv cg_n \pmod{m} \quad \text{iff} \quad n \geq n_0, \quad (4.27)$$

then the least positive integer s satisfying (4.27) is called **the constrained period of $\{g_n\}$ modulo m** and is denoted by $s = P'(m, g_n)$. The number c is called the **multiplier**.

Lemma 4.7: Let $\{w_n\}$ be the principal sequence in $\Omega(F_k(t))$, where $F_k(t)$ is denoted by (4.16). Then $P'(m, w_n) = s$ exists and the multiplier c is equal to w_{s+2k+1} (mod m). Furthermore, if $r = \text{ord}_m(c)$, then $P(m, w_n) = sr$, and the structure of $\{w_n \pmod{m}\}$ in a period is as follows:

$$\begin{cases} 0, \ \ldots, \ 0, \ 1, & w_{2k+2}, \quad w_{2k+3}, \quad \ldots, \ w_{s-1}, \\ 0, \ \ldots, \ 0, \ c, & cw_{2k+2}, \quad cw_{2k+3}, \quad \cdots, \ cw_{s-1}, \\ \cdots & \\ 0, \ \ldots, \ 0, \ c^{r-1}, & c^{r-1}w_{2k+2}, \ c^{r-1}w_{2k+3}, \ \ldots, \ c^{r-1}w_{s-1}. \end{cases} \quad (4.28)$$

Proof: Because $\{w_n\}$ is periodic, it must be constrained periodic [in the most special case, the multiplier c may be equal to 1 (mod m)]. We have $w_0 = \cdots = w_{2k} = 0$ and $w_{2k+1} = 1$. Replacing n by $2k+1$ in the expression

$$w_{n+s} \equiv cw_n \pmod{m}, \quad (4.29)$$

we obtain $c \equiv w_{s+2k+1} \pmod{m}$. By induction, from (4.29), we can get

$$w_{n+js} \equiv c^j w_n \pmod{m}. \quad (4.30)$$

If $j = r = \text{ord}_m(c)$, then (4.30) becomes $w_{n+rs} \equiv w_n \pmod{m}$. This means that $P(m, w_n) = sr$. In (4.30), let j be $0, 1, \ldots, r-1$ and n be $0, 1, \ldots, s-1$; then (4.28) follows. □

From Lemma 4.7, (4.18), and (3.1), we obtain

Theorem 4.4: Let $\{w_n\}$ be the principal sequence in $\Omega(F_k(t))$, where $F_k(t)$ is denoted by (4.16), and let p be an odd prime, $p > k$, $P'(p^m, w_n) = s$. If $w_n \equiv 0 \pmod{p^m}$ for $n \equiv i \pmod{s}$, then

$$u_n^{(k)} \equiv 0 \pmod{p^m} \text{ for } n \equiv i-k \pmod{s}$$

and

$$v_n^{(k+1)} \equiv 0 \pmod{p^m} \text{ for } n \equiv i-k \pmod{s} \text{ or } n \equiv 0 \pmod{p^m}.$$

Furthermore, if $\lambda p^r \equiv i-k \pmod{s}$, then $v_{\lambda p^r}^{(k+1)} \equiv 0 \pmod{p^{m+r}}$.

Example 1: Let $x=1, p=3$. Then $\Delta^2 = 5, p\nmid\Delta^2$. Hence, from (4.20), we obtain $P(3^m, f_n^{(k)}) = P(3^m, \ell_n^{(k)}) = 3^m \cdot P(3, f_n) = 8\cdot 3^m$ for $k=1,2$.

Example 2: Let $x=1, p=5$. Then $p|\Delta^2 = 5$. Hence, from (4.21), we get $P(5^m, f_n^{(k)}) = 4\cdot 5^{m+1}$ for $k = 2,3,4$, or $4\cdot 5^m$ for $k=1$ and, from (4.22), we get $P(5^m, \ell_n^{(k)}) = 4\cdot 5^{m+1}$ for $k=3,4$ or $4\cdot 5^m$ for $k=1,2$.

Example 3: We show that $f_n^{(2)} = 0 \pmod{10}$ iff $n \equiv 0, \pm 1, \pm 2 \pmod{25}$, and $\ell_n^{(3)} \equiv 0 \pmod{30}$ iff $n \equiv \pm 1, \pm 2 \pmod{25}$ or $n \equiv 0 \pmod 5$.

Proof of Example 3: We have $F_2(t) = (t^2-t-1)^3 = t^6 - 3t^5 + 5t^3 - 3t - 1 \equiv t^6 - 3t^5 - 3t - 1 \pmod 5$ for $x=1$. Let $\{w_n\}$ be the principal sequence in $\Omega(F_2(t))$. Then $w_{n+6} \equiv 3w_{n+5} + 3w_{n+1} + w_n \pmod 5$.

Calculate $\{w_n \pmod 5\}_0^\infty$ according to the last congruence:

$$0,0,0,0,0,1,-2,-1,2,1,1,-2,-1,2,1,2,1,-2,-1,2,-2,-1,2,1,-2,0,0,0,0,0,-2,\ldots \pmod 5.$$

This implies that $s = P'(5, w_n) = 25$ and $w_n \equiv 0 \pmod 5$ iff $n \equiv 0,1,2,3,4 \pmod{25}$. Hence, the example is proved by Theorem 4.1 and Theorem 4.4.

5. EVALUATION OF SOME SERIES INVOLVING $u_n^{(k)}$ AND $v_n^{(k)}$

Lemma 5.1:

$$1^\circ.\ \sum_{i=0}^{n} u_i = (u_{n+1} + u_n - 1)/x \quad (x \neq 0). \tag{5.1}$$

$$2^\circ.\ \sum_{i=0}^{n} v_i = (v_{n+1} + v_n - 2)/x + 1 \quad (x \neq 0). \tag{5.2}$$

$$3^\circ.\ \sum_{i=0}^{n} \binom{n}{i} x^i h_{i+r} = h_{2n+r}, \text{ where } \{h_n\} \text{ is } \{u_n\} \text{ or } \{v_n\}. \tag{5.3}$$

$$4^\circ.\ \sum_{i=0}^{n} (-1)^i \binom{n}{i} h_{2i+r} = (-1)^n x^n h_{n+r}, \text{ where } \{h_n\} \text{ is } \{u_n\} \text{ or } \{v_n\}. \tag{5.4}$$

$$5^\circ.\ \sum_{i=0}^{n} \binom{n}{i} u_{2i+r} = (x^2+4)^{n/2} u_{n+r} \text{ for } 2|n, \text{ or } (x^2+4)^{(n-1)/2} v_{n+r} \text{ for } 2\nmid n. \tag{5.5}$$

$$6^\circ.\ \sum_{i=0}^{n} \binom{n}{i} v_{2i+r} = (x^2+4)^{n/2} v_{n+r} \text{ for } 2|n, \text{ or } (x^2+r)^{(n+1)/2} u_{n+r} \text{ for } 2\nmid n. \tag{5.6}$$

Proof: We prove only 2° and 5°. The rest can be proved in the same way.

2°. $\sum_{i=0}^{n} v_i = \sum_{i=0}^{n}(\alpha^i+\beta^i) = (1-\alpha^{n+1})/(1-\alpha)+(1-\beta^{n+1})/(1-\beta)$

$$= (1-\alpha^{n+1}-\beta-\alpha^n+1-\beta^{n+1}-\alpha-\beta^n)/(-x) = (v_{n+1}+v_n-2)/x+1.$$

5°. We have

$$\sum_{i=0}^{n}\binom{n}{i}\alpha^{2i} = (1+\alpha^2)^n = (-\alpha\beta+\alpha^2)^n = \Delta^n\alpha^n.$$

For the same reason

$$\sum_{i=0}^{n}\binom{n}{i}\beta^{2i} = (-1)^n\Delta^n\beta^n.$$

Hence,

$$\sum_{i=0}^{n}\binom{n}{i}u_{2i+r} = \sum_{i=0}^{n}\binom{n}{i}(\alpha^{2i+r}-\beta^{2i+r})/\Delta = \Delta^n[\alpha^{n+r}-(-1)^n\beta^{n+r})/\Delta$$

$$= \Delta^n[\alpha^{n+r}-\beta^{n+r})/\Delta = (x^2+4)^{n/2}u_{n+r} \quad \text{for } 2\mid n,$$

$$\text{or} \qquad = \Delta^{n-1}(\alpha^{n+r}+\beta^{n+r}) = (x^2+4)^{(n-1)/2}v_{n+r} \quad \text{for } 2\nmid n. \qquad \square$$

Theorem 5.1:

$$\sum_{i=0}^{n} u_i^{(k)} = \sum_{i=0}^{k}(-1)^i(k)_i[u_{n+1}^{(k-i)}+u_n^{(k-i)}-\delta_{k,i}]/x^{i+1} \quad (x\neq 0); \tag{5.7}$$

$$\sum_{i=0}^{n} v_i^{(k)} = \sum_{i=0}^{k}(-1)^i(k)_i[v_{n+1}^{(k-i)}+v_n^{(k-i)}-2\delta_{k,i}]/x^{i+1} \quad (x\neq 0); \tag{5.8}$$

$$\sum_{i=0}^{n}\binom{n}{i}x^i h_{i+r}^{(k)} = \sum_{i=0}^{k}(-1)^i\binom{k}{i}(n)_i h_{2n-i+r}^{(k-i)}, \tag{5.9}$$

where $\{h_n^{(i)}\}$ is $\{u_n^{(i)}\}$ or $\{v_n^{(i)}\}$ $(i=0,\ldots,k)$;

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}h_{2i+r}^{(k)} = (-1)^n\sum_{i=0}^{k}\binom{k}{i}(n)_i x^{n-i}h_{n+r}^{(k-i)}, \tag{5.10}$$

where $\{h_n^{(i)}\}$ is $\{u_n^{(i)}\}$ or $\{v_n^{(i)}\}$ $(i=0,\ldots,k)$;

$$\sum_{i=0}^{n}\binom{n}{i}u_{2i+r}^{(k)} = \sum_{i=0}^{k}\binom{k}{i}u_{n+r}^{(k-i)}\frac{d^i}{dx^i}(x^2+4)^{n/2} \quad \text{for } 2\mid n,$$

$$\text{or} \qquad = \sum_{i=0}^{k}\binom{k}{i}v_{n+r}^{(k-i)}\frac{d^i}{dx^i}(x^2+4)^{(n-1)/2} \quad \text{for } 2\nmid n; \tag{5.11}$$

$$\sum_{i=0}^{n}\binom{n}{i}v_{2i+r}^{(k)} = \sum_{i=0}^{k}\binom{k}{i}v_{n+r}^{(k-i)}\frac{d^i}{dx^i}(x^2+4)^{n/2} \quad \text{for } 2\mid n,$$

$$\text{or} \qquad = \sum_{i=0}^{k}\binom{k}{i}u_{n+r}^{(k-i)}\frac{d^i}{dx^i}(x^2+r)^{(n+1)/2} \quad \text{for } 2\nmid n. \tag{5.12}$$

Proof: Every one of (5.7), (5.8), (5.10)-(5.12) can be proved straightforwardly by differentiating the corresponding one of (5.1), (5.2), (5.4)-(5.6). The proof of (5.9) is as follows.

Let

$$g_{n,k,r} = g_{n,k,r}(x) = \sum_{i=0}^{k} \binom{n}{i} x^i h_{i+r}^{(k)}. \tag{5.13}$$

Then

$$g'_{n,k,r} = \sum_{i=0}^{k} \binom{n}{i} x^i h_{i+r}^{(k+1)} + \sum_{i=1}^{n} \binom{n}{i} i x^{i-1} h_{i+r}^{(k)} = g_{n,k+1,r} + n \cdot g_{n-1,k,r+1}.$$

So

$$g_{n,k+1,r} = g'_{n,k,r} - n \cdot g_{n-1,k,r+1}. \tag{5.14}$$

When $k=0$, from (5.3), we can see that (5.9) holds. Assume that (5.9) holds for k; then from (5.14), we have

$$g_{n,k+1,r} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (n)_i h_{2n-i+r}^{(k+1-i)} - n \sum_{i=0}^{k} (-1)^i \binom{k}{i} (n-1)_i h_{2n-1-i+r}^{(k-i)}.$$

The second summation in the right side of the last expression can be rewritten as

$$-n \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (n-1)_i h_{2n-1-i+r}^{(k-i)} - n(-1)^k \cdot (n-1)_k h_{2n-1-k+r}$$

$$= \sum_{i=1}^{k} (-1)^i \binom{k}{i-1} (n)_i h_{2n-i+r}^{(k+1-i)} + (-1)^{k+1} (n)_{k+1} h_{2n-(k+1)+r}.$$

From this, it follows that

$$g_{n,k+1,r} = \sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} (n)_i h_{2n-i+r}^{(k+1-i)},$$

that is, (5.9) also holds for $k+1$, and we are done. □

It is known that the generating function of $\{u_n^{(k)}\}$ is expressed by (2.7). It is well known that the generating function of $\{v_n\}$ is

$$V(t) = (2-xt)/(1-xt-t^2). \tag{5.15}$$

Differentiating (5.15), we can know that the generating function of $\{v_n^{(k)}\}$ is

$$V_k(t) = k!\,t^k(1+t^2)/(1-xt-t^2)^{k+1} \quad (k \ge 1). \tag{5.16}$$

Obviously, the following identities hold:

$$U_k(t) \cdot U_r(t) = \frac{k!\,r!}{(k+r+1)!} U_{k+r+1}(t);$$

$$V_k(t) \cdot V_r(t) = \frac{k!\,r!}{(k+r+1)!} (t+t^{-1}) V_{k+r+1}(t) \quad (k, r \ge 1);$$

$$U_k(t) \cdot V_r(t) = \frac{k!\,r!}{(k+r+1)!} V_{k+r+1}(t) \quad (r \ge 1);$$

$$U_k(t) \cdot V(t) = \frac{1}{k+1} (2t^{-1} - x) U_{k+1}(t);$$

$$V_k(t)\cdot V(t)=\frac{1}{k+1}(2t^{-1}-x)V_{k+1}(t)\quad (k\geq 1).$$

Equalizing the coefficients of t^n of the two sides in each of the above identities, we have

Theorem 5.2:

$$\sum_{i=0}^{n}u_i^{(k)}u_{n-i}^{(r)}=\frac{k!r!}{(k+r+1)!}u_n^{(k+r+1)};\tag{5.17}$$

$$\sum_{i=0}^{n}v_i^{(k)}v_{n-i}^{(r)}=\frac{k!r!}{(k+r+1)!}(v_{n-i}^{(k+r+1)}+v_{n+1}^{(k+r+1)})\quad (k,r\geq 1);\tag{5.18}$$

$$\sum_{i=0}^{n}u_i^{(k)}v_{n-i}^{(r)}=\frac{k!r!}{(k+r+1)!}v_n^{(k+r+1)}\quad (r\geq 1);\tag{5.19}$$

$$\sum_{i=0}^{n}u_i^{(k)}v_{n-i}=\frac{1}{k+1}(2u_{n+1}^{(k+1)}-xu_n^{(k+1)});\tag{5.20}$$

$$\sum_{i=0}^{n}v_i^{(k)}v_{n-i}=\frac{1}{k+1}(2v_{n+1}^{(k+1)}-xv_n^{(k+1)})\quad (k\geq 1).\tag{5.21}$$

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