ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to Fibonacci@MathPro.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \ F_0 = 0, \ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \ L_0 = 2, \ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-814 (Corrected) ***Proposed by M. N. Deshpande, Institute of Science, Nagpur, India***

Show that, for each positive integer n, there exists a constant C_n such that $F_{2n+2i}F_{2i} + C_n$ and $F_{2n+2i+1}F_{2i+1} - C_n$ are both perfect squares for all positive integers i.

B-821 ***Proposed by L. A. G. Dresel, Reading, England***

Consider the rectangle with sides of lengths F_{n-1} and F_{n+1}. Let A_n be its area, and let d_n be the length of its diagonal. Prove that $d_n^2 = 3A_n \pm 1$.

B-822 ***Proposed by Anthony Sofo, Victoria University of Technology, Australia***

For $n > 0$, simplify $\sqrt[n]{\alpha F_n + F_{n-1}} + (-1)^{n+1}\sqrt[n]{F_{n+1} - \alpha F_n}$.

B-823 ***Proposed by Pentti Haukkanen, University of Tampere, Finland***

It is easy to see that the solution of the recurrence relation $A_{n+2} = -A_{n+1} + A_n$, $A_0 = 0$, $A_1 = 1$, can be written as $A_n = (-1)^{n+1}F_n$.

Find a solution to the recurrence $A_{n+2} = -A_{n+1} + A_n$, $A_0 = 1$, $A_1 = 1$, in terms of F_n and L_n.

B-824 ***Proposed by Brian D. Beasley, Presbyterian College, Clinton, SC***

Fix a nonnegative integer m. Solve the recurrence $A_{n+2} = L_{2m+1}A_{n+1} + A_n$, for $n \ge 0$, with initial conditions $A_0 = 1$ and $A_1 = L_{2m+1}$, expressing your answer in terms of the Fibonacci and/or Lucas numbers.

B-825 ***Proposed by Lawrence Somer, The Catholic Univ. of America, Washington, D.C.***

Let $\langle V_n \rangle$ be a sequence defined by the recurrence $V_n = PV_{n+1} - QV_n$, where P and Q are integers and $V_0 = 2$, $V_1 = P$. The integer d is said to be a divisor of $\langle V_n \rangle$ if $d \mid V_n$ for some $n \geq 1$.

(a) If P and Q are both even, show that 2^m is a divisor of $\langle V_n \rangle$ for any $m \geq 1$.

(b) If P or Q is odd, show that there exists a fixed nonnegative integer k such that 2^k is a divisor of $\langle V_n \rangle$ but 2^{k+1} is not a divisor of $\langle V_n \rangle$. If exactly one of P or Q is even, show that $2^k \mid V_1$; if P and Q are both odd, show that $2^k \mid V_3$.

NOTE: The Elementary Problems Column is in need of more *easy*, yet elegant and nonroutine problems.

SOLUTIONS

Double, Double, Triangular Numbers and Trouble

B-802 ***Proposed by Al Dorp, Edgemere, NY***
(Vol. 34, no. 1, February 1996)

Let $T_n = n(n+1)/2$ denote the n^{th} triangular number. Find a formula for T_{2n} in terms of T_n.

Editorial Note: I loved this problem because of all the varied solutions. It is amazing how resourceful our readers can be, and I enjoyed seeing some ingenious solutions.

Solutions by various solvers

Many solvers: $T_{2n} = 4T_n - n$.

Marjorie Bicknell-Johnson: $T_{2n} = n^2 + 2T_n$.

Many solvers: $T_{2n} = \dfrac{8T_n + 1 - \sqrt{8T_n + 1}}{2}$.

Joseph J. Koštál: $T_{2n} = n\sqrt{8T_n + 1}$.

Paul S. Bruckman: $T_{2n} = \dbinom{\sqrt{8T_n + 1}}{2}$.

Herta T. Freitag: $T_{2n} = \frac{1}{2}n^2(n+1)(2n+1)/T_n$.

Many solvers: $T_{2n} = 3T_n + T_{n-1}$.

David Zeitlin: $T_{2n} = 6T_n - 3T_{n+1} + T_{n+2}$.

One reader came up with the wondrous formula $T_{2n} = (T_n^2 + 101T_n - 12)/30$. *Unfortunately, this formula was too good to be true.*

Generalization by David Terr, University of California, Berkeley, CA, and Daina A. Krigens, Encinatas, CA (independently)

$$T_{kn} = k^2 T_n - nT_{k-1}.$$

Generalization by Marjorie Bicknell-Johnson, Santa Clara, CA

$$T_{kn+p} = [(2p+1)k - 4T_p]T_n + (n+1)^2 T_p + n^2 T_{k-p-1}.$$

Also solved by Mohammad K. Azarian, Brian D. Beasley, Charles K. Cook, M. N. Deshpande, Leonard A. G. Dresel, Steve Edwards, Russell Euler, Thomas M. Green, Russell Jay Hendel, Gerald Heuer, Harris Kwong, Carl Libis, Bob Prielipp, John A. Schumaker, R. P. Sealy, H.-J. Seiffert, Lawrence Somer, and the proposer.

Half a Lucas Sum

B-803 ***Proposed by Herta Freitag, Roanoke, VA (Vol. 34, no. 1, February 1996)***

For n even and positive, evaluate

$$\sum_{i=0}^{n/2}\binom{n}{i}L_{n-2i}.$$

Solution by L. A. G. Dresel, Reading, England

Let $A=\sum_{i=0}^{n}\binom{n}{i}L_{n-2i}$ and $B=\sum_{i=0}^{n/2}\binom{n}{i}L_{n-2i}$. Since $\binom{n}{n-i}=\binom{n}{i}$ and $L_{-2t}=L_{2t}$ and n is even, we have $A+\binom{n}{n/2}L_0=2B$, and

$$A=\sum_{i=0}^{n}\binom{n}{i}(\alpha^{n-2i}+\beta^{n-2i})=\alpha^{-n}(\alpha^2+1)^n+\beta^{-n}(\beta^2+1)^n=(\alpha+\alpha^{-1})^n+(\beta+\beta^{-1})^n.$$

But $\alpha+\alpha^{-1}=\sqrt{5}$ and $\beta+\beta^{-1}=-\sqrt{5}$. Therefore, $A=2(5^{n/2})$ and $B=5^{n/2}+\binom{n}{n/2}$.

Haukkanen found a corresponding formula for Fibonacci numbers:

$$\sum_{i=0}^{(n-1)/2}\binom{n}{i}F_{n-2i}=5^{(n-1)/2},\ n\text{ odd}.$$

Seiffert gave the generalization,

$$\sum_{i=0}^{n/2}(-1)^{(k+1)i}\binom{n}{i}L_{k(n-2i)}=5^{n/2}F_k^n+(-1)^{(k+1)n/2}\binom{n}{n/2},$$

which comes from [1] and can also be traced back to Lucas [3]. Haukkanen found a generalization for the sequences defined by $U_n=mU_{n-1}+U_{n-2}$, $U_0=0$, $U_1=1$, *and* $V_n=mV_{n-1}+V_{n-2}$, $V_0=2$, $V_1=m$, *which comes from [2]:*

$$\sum_{i=0}^{n/2}\binom{n}{i}V_{k(n-2i)}=(m^2+4)^{n/2}U_k^n+\binom{n}{n/2},\ k\text{ odd}, n\text{ even}.$$

References

1. The Citadel Problem Solving Group. "Problem 519: A Linear Combination of Lucas Numbers." *The College Mathematics Journal* **26.1** (1995):70.
2. P. Filipponi. "Waring's Formula, the Binomial Formula, and Generalized Fibonacci Matrices." *The Fibonacci Quarterly* **30.3** (1992):225-31.
3. Edouard Lucas. *The Theory of Simply Periodic Numerical Functions.* Santa Clara, CA: The Fibonacci Association, 1969.

Also solved by Paul Bruckman, M. N. Deshpande, Russell Euler, Pentti Haukkanen, Russell Jay Hendel, R. P. Sealy, H.-J. Seiffert, David Zeitlin, and the proposer.

Finding an Identity without a Crystal Ball

B-804 ***Proposed by the editor***
(Vol. 34, no. 1, February 1996)

Find integers a, b, c, and d (with $1 < a < b < c < d$) that make the following an identity:

$$F_n = F_{n-a} + 9342F_{n-b} + F_{n-c} + F_{n-d}.$$

Solution by L. A. G. Dresel, Reading, England

We note that $9342 = 9349 - 7 = L_{19} - L_4$. Using the identities I_{23} and I_{21} of [1], we have

$$F_{m+19} - F_{m-19} = F_m L_{19} \quad \text{and} \quad F_{m+4} + F_{m-4} = F_m L_4.$$

Subtracting, we obtain $F_{m+19} - F_{m-19} - F_{m+4} - F_{m-4} = F_m(L_{19} - L_4)$. Putting $n = m+19$ and rearranging, gives the identity $F_n = F_{n-15} + 9342F_{n-19} + F_{n-23} + F_{n-38}$. We conclude that $a = 15$, $b = 19$, $c = 23$, and $d = 38$ provides a solution to this problem.

Bruckman, Dresel, Johnson, and Seiffert all found the general identity

$$F_n = F_{n-u+v} + (L_u - L_v)F_{n-u} + F_{n-u-v} + F_{n-2u}$$

with u odd and v even. Zeitlin found the identity $F_n = F_{n-2} + 9349F_{n-20} + F_{n-40} + F_{n-41}$.

Reference

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, CA: The Fibonacci Association, 1979.

Also solved by Paul S. Bruckman, Russell Jay Hendel, Marjorie Bicknell-Johnson, Daina A. Krigens, H.-J. Seiffert, David Zeitlin, and the proposer.

A Slightly Perturbed Fibonacci Sequence

B-805 ***Proposed by David Zeitlin, Minneapolis, MN***
(Vol. 34, no. 1, February 1996)

Solve the recurrence $P_{n+6} = P_{n+5} + P_{n+4} - P_{n+2} + P_{n+1} + P_n$, for $n \geq 0$, with initial conditions $P_0 = 1, P_1 = 1, P_2 = 2, P_3 = 3, P_4 = 4,$ and $P_5 = 7$.

Editorial composite of the solutions received:

Let
$$A_n = \begin{cases} 0, & \text{if } n \equiv 1, 5 \pmod{8}, \\ 1, & \text{if } n \equiv 0, 2, 3 \pmod{8}, \\ -1, & \text{if } n \equiv 4, 6, 7 \pmod{8}. \end{cases}$$

Then note that A_n satisfies the given recurrence. Also note that F_n satisfies the given recurrence. Thus, any linear combination of F_n and A_n satisfies the given recurrence. Now, one only needs to find the linear combination that meets the initial conditions. The solution is $P_n = (A_n + F_{n+3})/3$.

This can also be written in the form $P_n = \left\lfloor \frac{1+F_{n+3}}{3} \right\rfloor$.

Also solved by Brian D. Beasley, Paul Bruckman, Leonard A. G. Dresel, Pentti Haukkanen, Russell Jay Hendel, Gerald A. Heuer, Harris Kwong, H.-J. Seiffert, and the proposer. A partial solution was obtained by Jackie Roehl.

Power Series with Fibonacci Features

B-806 ***Proposed by Mohammad K. Azarian, University of Evansville, Evansville, IN (Vol. 34, no. 1, February 1996)***

(a) Show that the coefficient of every term in the expansion of $\frac{x}{1-2x+x^3}$ is the difference of two Fibonacci numbers.

(b) Show that the coefficient of every term in the expansion of $\frac{x}{1-2x-2x^2+x^3}$ is the product of two consecutive Fibonacci numbers.

Solution by Harris Kwong, SUNY College at Fredonia, Fredonia, NY

(a) It is well known that $\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2}$. Hence,

$$\frac{x}{1-2x+x^3} = \frac{x}{1-x-x^2} \cdot \frac{1}{1-x} = \left(\sum_{n=0}^{\infty} F_n x^n\right)\left(\sum_{n=0}^{\infty} x^n\right)$$

generates $c_n = \sum_{k=0}^{n} F_k = F_{n+2} - F_2$.

(b) Routine calculation reveals that

$$\frac{x}{1-3x+x^2} = \frac{x}{(1-\alpha^2 x)(1-\beta^2 x)} = \sum_{n=0}^{\infty}\left(\frac{\alpha^{2n}-\beta^{2n}}{\alpha^2-\beta^2}\right)x^n = \sum_{n=0}^{\infty} F_{2n} x^n.$$

Thus,

$$\frac{x}{1-2x-2x^2+x^3} = \frac{x}{1-3x+x^2} \cdot \frac{1}{1+x} = \left(\sum_{n=0}^{\infty} F_{2n} x^n\right)\left(\sum_{n=0}^{\infty} (-1)^n x^n\right)$$

generates $d_n = F_{2n} - F_{2n-2} + \cdots + (-1)^n F_0$, which satisfies the recurrence relation $d_{n+1} + d_n = F_{2n+2}$. Using $F_{2n+2} = L_{n+1}F_{n+1} = (F_{n+2} + F_n)F_{n+1} = F_{n+1}F_{n+2} + F_n F_{n+1}$ and induction, we find that $d_n = F_n F_{n+1}$.

Also solved by Paul S. Bruckman, Charles K. Cook, M. N. Deshpande, Steve Edwards, Russell Euler, Pentti Haukkanen, Russell Jay Hendel, Carl Libis, H.-J. Seiffert, David Zeitlin, and the proposer.

Generalized Mod Squad

B-807 ***Proposed by R. André-Jeannin, Longwy, France (Vol. 34, no. 1, February 1996)***

The sequence $\langle W_n \rangle$ is defined by the recurrence $W_n = PW_{n-1} - QW_{n-2}$, for $n \geq 2$, with initial conditions $W_0 = a$ and $W_1 = b$, where a and b are integers and P and Q are odd integers. Prove that, for $k \geq 0$, $W_{n+3\cdot 2^k} \equiv W_n \pmod{2^{k+1}}$.

Solution by Lawrence Somer, The Catholic Univ. of America, Washington, D.C.

Let $\langle U_n \rangle$ and $\langle V_n \rangle$ be sequences satisfying the same recursion as $\langle W_n \rangle$ with initial terms $U_0 = 0$, $U_1 = 1$, and $V_0 = 2, V_1 = P$, respectively. It can be proved by the Binet formulas and induction that

$$W_{n+m} = -QW_{n-1}U_m + W_n U_{m+1}, \tag{1}$$

$$U_{2n} = U_n V_n, \tag{2}$$

and

$$U_{2n+1} = -QU_n^2 + U_{n+1}^2. \tag{3}$$

Since P and Q are odd, the sequence $\langle U_n \rangle$, modulo 2, is 0, 1, 1, 0, 1, 1, ... By inspection, one sees that both $\langle U_n \rangle$ and $\langle V_n \rangle$ are purely periodic modulo 2 with periods equal to 3. Moreover, one sees by inspection that $U_n \equiv 0 \pmod 2$ if and only if $3|n$, and $V_n \equiv 0 \pmod 2$ if and only if $3|n$.

We now show by induction that $U_{3\cdot 2^k} \equiv 0 \pmod{2^{k+1}}$ for $k \geq 0$. If $k = 0$, one sees by what was stated above that $U_3 \equiv 0 \pmod 2$. Suppose the result is true up to k. Then

$$U_{3\cdot 2^{k+1}} = U_{3\cdot 2^k} V_{3\cdot 2^k} \tag{4}$$

by formula (2). Since $U_{3\cdot 2^k} \equiv 0 \pmod{2^{k+1}}$ by our induction hypothesis, and $V_{3\cdot 2^k} \equiv 0 \pmod 2$, we see from (4) that $U_{3\cdot 2^{k+1}} \equiv 0 \pmod{2^{k+2}}$, and our induction is complete.

We next show by induction that $U_{3\cdot 2^k+1} \equiv 1 \pmod{2^{k+1}}$ for $k \geq 0$. If $k = 0$, then, by what was stated earlier, $U_4 \equiv 1 \pmod 2$. Assume the result is true up to k. Then

$$U_{3\cdot 2^{k+1}+1} = -QU_{3\cdot 2^k}^2 + U_{3\cdot 2^k+1}^2 \tag{5}$$

by formula (3). Since $U_{3\cdot 2^k} \equiv 0 \pmod{2^{k+1}}$ by our above result, $U_{3\cdot 2^k}^2 \equiv 0 \pmod{2^{k+2}}$. Since $U_{3\cdot 2^k+1} = 1 + r2^{k+1}$ for some integer r by our induction hypothesis, we see that $U_{3\cdot 2^k+1}^2 \equiv 1 \pmod{2^{k+2}}$. Then, by (5), we have $U_{3\cdot 2^{k+1}+1} \equiv 1 \pmod{2^{k+2}}$ and our induction is complete.

We now see that

$$W_{n+3\cdot 2^k} = -QW_{n-1}U_{3\cdot 2^k} + W_n U_{3\cdot 2^k+1} \equiv (-QW_{n-1})\cdot 0 + W_n \cdot 1 \equiv W_n \pmod{2^{k+1}}.$$

The proposer mentions that this problem came about by his efforts to generalize problem B-732.

Also solved by Paul S. Bruckman, Leonard A. G. Dresel, Harris Kwong, H.-J. Seiffert, and the proposer.

Addenda: The editor wishes to apologize for misplacing some solutions that were sent in on time. We therefore acknowledge solutions from the following solvers:

Brian D. Beasley—B-790, 791;
L. A. G. Dresel—B-796, 797, 798, 799, 801;
C. Georghiou—B-796, 797, 798, 799, 800, 801;
Pentti Haukkanen—B-793;
Russell J. Hendel—B-784;
Gerald A. Heuer—B-792, 793;
Igor O. Popov—B-796, 797;
Dorka O. Popova—B-799.

Errata: In the solution to B-773 (Feb. 1996), identity (I_{11}) should read $F_n^2 + F_{n-1}^2 = F_{2n-1}$. In the solution to B-798 (Nov. 1996), the comment by Bloom should read: If p is odd, $k \geq 1$, and F_n is divisible by p^k but not by p^{k+1}, then F_{np} is divisible by p^{k+1} but not by p^{k+2}.

David Zeitlin

I have been informed by David Zeitlin's niece that Dr. Zeitlin passed away on Nov. 5, 1996. She wrote that her uncle's entire life was devoted to mathematics. Readers of this column are no doubt familiar with his writings, since he has been an active contributor to this column and to this journal since it began back in 1963. His papers have been way ahead of their time. Readers who go back to some of the early issues of this Quarterly will find some amazing results buried deep within his papers. I will miss his scrawled handwriting and the incredible formulas that arrive in the mail in response to some of the problems in this column.

—Editor