AN OBSERVATION ON SUMMATION FORMULAS FOR GENERALIZED SEQUENCES

Piero Filipponi

Fondazione Ugo Bordoni, Via B. Castiglione 59, I-00142 Rome, Italy
e-mail: filippo@fub.it

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1. PRELIMINARIES

For a, b, p, and q arbitrary integers, in the notation of Horadam [2] write

$$W_n = W_n(a,b;\, p,q) \tag{1.1}$$

so that

$$W_0 = a,\ W_1 = b,\ W_n = pW_{n-1} - qW_{n-2} \quad \text{for } n \geq 2. \tag{1.2}$$

In particular, we write

$$\begin{cases} U_n = W_n(0,1;\, p,q), \\ V_n = W_n(2,p;\, p,q). \end{cases} \tag{1.3}$$

The Binet forms for U_n and V_n are

$$U_n = (\alpha^n - \beta^n)/\sqrt{\Delta}, \tag{1.4}$$

$$V_n = \alpha^n + \beta^n, \tag{1.5}$$

where

$$\Delta \doteq p^2 - 4q, \tag{1.6}$$

and

$$\alpha = (p+\sqrt{\Delta})/2 \quad \text{and} \quad \beta = (p-\sqrt{\Delta})/2 \tag{1.7}$$

are the roots, assumed distinct, of the equation $x^2 - px + q = 0$. Observe that (1.7) yields the two identities

$$\alpha + \beta = p \quad \text{and} \quad \alpha\beta = q. \tag{1.8}$$

As done in [3], throughout this note it is assumed that

$$\Delta > 0, \tag{1.9}$$

so that α, β, and $\sqrt{\Delta}$ are real and $\alpha \neq \beta$. We also assume that

$$q \neq 0 \tag{1.10}$$

to warrant that (1.2) is a second-order recurrence relation. Finally, observe that the particular case $p = 0$ yields

$$U_n = \begin{cases} 0 & (n \text{ even}), \\ (-q)^{(n-1)/2} & (n \text{ odd}), \end{cases} \quad \text{and} \quad V_n = \begin{cases} 2(-q)^{n/2} & (n \text{ even}), \\ 0 & (n \text{ odd}). \end{cases} \tag{1.11}$$

Throughout our discussion, the special sequences (1.11) will not be considered, that is, we shall assume that

$$p \neq 0. \tag{1.12}$$

2. MOTIVATION OF THIS NOTE

Some months ago, I had the opportunity of reviewing (for the American Mathematical Society) an article [3] in which the author establishes several summation formulas for U_n and V_n by using the Binet forms (1.4) and (1.5) and the geometric series formula (g.s.f.).

As usual, I began my review by checking the results numerically. Without intention, I chose the values $p=4$ and $q=3$ which satisfy (1.9), (1.10), and (1.12) and, to my great surprise, noticed that the formulas in [3] do not work for these values of p and q because certain denominators vanish. On the other hand, I ascertained that they work perfectly for many other values of these parameters.

The aim of this note is to bring to the attention of the reader a fact that seems to have passed unnoticed in spite of its simplicity: if $q=p-1$, then either α or β [see (1.7)] equals 1, whereas if $q=-(p+1)$, then either α or β equals -1. Consequently, for obtaining summation formulas for U_n and V_n, the g.s.f. must be used *properly* to avoid getting meaningless expressions.

The example given in Section 4 will clarify our statement.

3. BINET FORMS FOR U_n AND V_n IN THE SPECIAL CASES $q=p-1$ AND $q=-(p+1)$

The Binet forms for U_n and V_n in the cases $q=p-1$ and $q=-(p+1)$ obviously play a crucial role throughout our discussion.

3.1 The case $q=p-1$

If

$$q=p-1 \tag{3.1}$$

then the expression (1.6) becomes

$$\Delta=p^2-4p+4 \tag{3.2}$$

whence, to fulfill (1.9), we must impose the condition

$$p\neq 2. \tag{3.3}$$

Remark 1: Conditions (3.1), (1.12), and (3.3) imply that

$$q\neq \pm 1. \tag{3.4}$$

Since we assumed that $\sqrt{\Delta}$ is positive [see (1.9)], (3.2) also implies that

$$\sqrt{\Delta}=\begin{cases} p-2, & \text{if } p>2,\\ 2-p, & \text{if } p<2,\end{cases} \tag{3.5}$$

whence [see (1.7)]

$$\alpha=\begin{cases} p-1=q \text{ (and } \beta=1), & \text{if } p>2,\\ 1 \text{ (and } \beta=q), & \text{if } p<2.\end{cases} \tag{3.6}$$

From (1.4), (1.5), (3.6), (3.5), and (3.1), it can be seen readily that the Binet forms for U_n and V_n are

$$U_n = \frac{q^n - 1}{q-1} \quad \text{[cf. (3.4)]} \tag{3.7}$$

and

$$V_n = q^n + 1. \tag{3.8}$$

Remark 2: By virtue of condition (1.10), the Binet forms (3.7) and (3.8) also have meaning for negative values of n.

3.2 The Case $q = -(p+1)$

If

$$q = -(p+1), \tag{3.9}$$

then expression (1.6) becomes

$$\Delta = p^2 + 4p + 4 \tag{3.10}$$

whence, to fulfill (1.9), we must impose the condition

$$p \neq -2 \tag{3.11}$$

which, due to (3.9) and (1.12), implies (3.4) as well.

Since we assumed that $\sqrt{\Delta}$ is positive, (3.10) also implies that

$$\sqrt{\Delta} = \begin{cases} p+2, & \text{if } p > -2, \\ -(p+2), & \text{if } p < -2, \end{cases} \tag{3.12}$$

whence [see (1.7)]

$$\alpha = \begin{cases} p+1 = -q \text{ (and } \beta = -1), & \text{if } p > -2, \\ -1 \text{ (and } \beta = -q), & \text{if } p < -2. \end{cases} \tag{3.13}$$

From (1.4), (1.5), (3.13), (3.12), and (3.9), it can be seen readily that the Binet forms for U_n and V_n are

$$U_n = (-1)^n \frac{q^n - 1}{1-q} \quad \text{[cf. (3.4)]}, \tag{3.14}$$

and

$$V_n = (-1)^n (q^n + 1). \tag{3.15}$$

Observe that Remark 2 also applies to the Binet forms (3.14) and (3.15).

4. SUMMATION FORMULAS THAT DO NOT HAVE GENERAL VALIDITY

Here we clarify the malfunctioning of the summation formulas in [3] by means of the following example. By using (1.5) and the g.s.f. {*without realizing that, if $q = p-1$, then α (or β) $= 1$, and if $q = -(p+1)$*, then α (*or* β) $= -1$ [see (3.6) and (3.13), respectively]}, after some simple manipulation involving the use of (1.8), one gets

$$\sum_{k=0}^{n} V_{km+r} = \frac{q^m (V_{mn+r} - V_{r-m}) + V_r - V_{m(n+1)+r}}{q^m - V_m + 1} \quad (m \neq 0). \tag{4.1}$$

Remark 3: The right-hand side of (4.1) may involve the use of the extension

$$V_{-m} = V_m / q^m, \tag{4.2}$$

which can be obtained immediately from (1.8).

Warning: Formula (4.1) works for all values of p and q except for those values for which either (3.1) (m arbitrary) or (3.9) (m even) holds. In fact, in these cases, from (3.8) [or (3.15)] we have $q^m - V_m + 1 = 0$. More precisely, it can be proved that the right-hand side of (4.1) assumes the indeterminate form 0/0. Analogous summation formulas yield the same indeterminate form.

If (3.1) holds, the correct closed-form expression for the left-hand side of (4.1) is

$$\begin{aligned}\sum_{k=0}^{n} V_{km+r} &= \sum_{k=0}^{n} (q^{km+r}+1) \quad \text{[from (3.8)]}\\ &= n+1+q^r \frac{q^{m(n+1)}-1}{q^m-1} = n+1+q^r \frac{V_{m(n+1)}-2}{V_m-2} \quad (m \neq 0).\end{aligned} \tag{4.3}$$

If (3.9) holds and m is even, from (3.15), the correct closed-form expression for the left-hand side of (4.1) is readily found to be

$$\sum_{k=0}^{n} V_{km+r} = (-1)^r(n+1) + (-q)^r \frac{V_{m(n+1)}-2}{V_m-2} \quad (m \neq 0, \text{ even}). \tag{4.4}$$

Observe that, if (3.9) holds and m is odd, the expression

$$\sum_{k=0}^{n} V_{km+r} = \begin{cases} (-1)^r + (-q)^r V_{m(n+1)} / V_m & (n \text{ even}),\\ (-q)^r (V_{m(n+1)} - 2) / V_m & (n \text{ odd}),\end{cases} \tag{4.5}$$

obtainable from (3.15), is nothing but a compact form for expression (4.1) which, in this case, works as well.

5. SUMMATION FORMULAS FOR U_n AND V_n WHEN $q = p-1$

We conclude this note by giving a brief account of the various kinds of summation formulas for U_n and V_n that are valid when (3.1) and (3.4) hold. Since their proofs are straightforward, they are omitted for brevity. We confine ourselves to mentioning that the proofs of (5.4)-(5.5) and (5.6)-(5.7) involve the use of the identities—see (3.1) and (3.4) of [1]—

$$\sum_{i=0}^{h} iy^i = \frac{hy^{h+2}-(h+1)y^{h+1}+y}{(y-1)^2} \quad \text{and} \quad \sum_{i=0}^{h} \binom{h}{i} iy^i = hy(y+1)^{h-1},$$

respectively.

$$\sum_{k=0}^{n} U_{km+r} = \frac{q^r U_{m(n+1)}}{(q-1)U_m} - \frac{n+1}{q-1} \quad (m \neq 0), \tag{5.1}$$

$$\sum_{k=0}^{n} \binom{n}{k} U_{km+r} = \frac{q^r V_m^n - 2^n}{q-1}, \tag{5.2}$$

$$\sum_{k=0}^{n} \binom{n}{k} V_{km+r} = q^r V_m^n + 2^n, \tag{5.3}$$

$$\sum_{k=0}^{n} kU_{km+r} = q^r \frac{nU_{m(n+2)} - (n+1)U_{m(n+1)} + U_m}{[(q-1)U_m]^2} - \frac{n(n+1)}{2(q-1)} \quad (m \neq 0), \tag{5.4}$$

$$\sum_{k=0}^{n} kV_{km+r} = q^r \frac{nV_{m(n+2)} - (n+1)V_{m(n+1)} + V_m}{[(q-1)U_m]^2} - \frac{n(n+1)}{2} \quad (m \neq 0), \tag{5.5}$$

$$\sum_{k=0}^{n} k\binom{n}{k} U_{km+r} = \frac{n}{q-1}(q^{m+r}V_m^{n-1} - 2^{n-1}), \tag{5.6}$$

$$\sum_{k=0}^{n} k\binom{n}{k} V_{km+r} = n(q^{m+r}V_m^{n-1} + 2^{n-1}). \tag{5.7}$$

It is obvious that summations (5.1)-(5.7) can be expressed simply in terms of powers of q. Doing so, we sometimes obtain more compact expressions. For example, we get

$$\sum_{k=0}^{n} kV_{km+r} = q^{m+r} \frac{q^{mn}[n(q^m - 1) - 1] + 1}{(q^m - 1)^2} + \frac{n(n+1)}{2} \quad (m \neq 0). \tag{5.5'}$$

Finally, we give the following example pertaining to alternate sign summations:

$$\sum_{k=0}^{n} k\binom{n}{k}(-1)^k V_{km+r} = \begin{cases} 0, & \text{if } n = 0, \\ -V_{m+r}, & \text{if } n = 1, \\ n(-1)^n q^{m+r}[(q-1)U_m]^{n-1}, & \text{if } n > 1. \end{cases} \tag{5.8}$$

The interested reader is urged to work out analogous summation formulas for the case in which $q = -(p+1)$ and m is even.

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