THE APPEARANCE OF FIBONACCI AND LUCAS NUMBERS IN THE SIMULATION OF ELECTRICAL POWER LINES SUPPLIED BY TWO SIDES

Giuseppe Ferri
Dipartimento di Ingegneria Elettrica-Facoltá di Ingegneria, Universitá di L'Aquila
Localitá Monteluco di Roio, 67040 Poggio di Roio, L'Aquila, Italia
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INTRODUCTION

In the analysis of some physical structures, the possibility of modeling them with an electrical circuit is particularly important because it allows the determination of the characteristic behavior by means of a simple circuital analysis. Moreover, it is also interesting to have a different method of measurement evaluation, comparable with the "direct" one, which sometimes either is not simple or requires the use of computer programs which on some occasions do not go into convergence. Finally, it can make a contribution to the mathematical interest in testing of network software algorithms for solving linear equation systems.

In this article, a symmetrical ladder network is used as a model for the simulation of electrical power lines. Fibonacci and Lucas numbers come out from the analysis of the power distribution among the users. The electrical characteristics of the ladder network have also been determined in a closed form using a theory previously developed by the author [1].

1. MODELING OF A POWER ELECTRIC LINE

Let us consider a high voltage electric line, supplied by the two sides, which gives power to users distributed along the line, as in Figure 1.

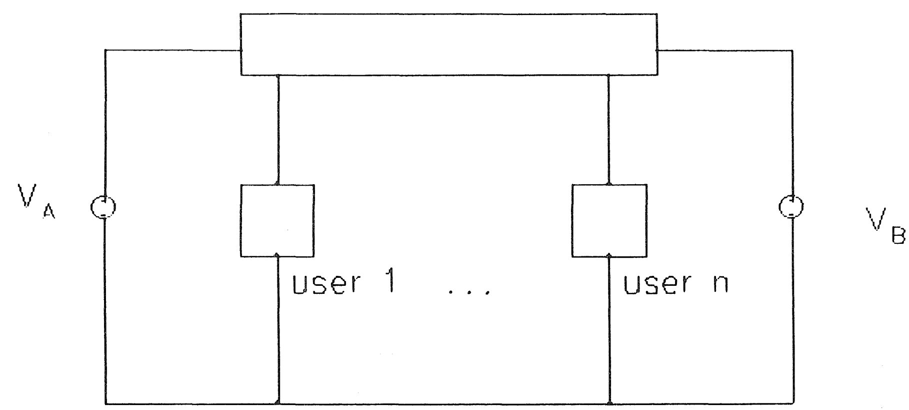

FIGURE 1. The Electrical Power Line Supplied by Two Sides

A ladder structure (Fig. 2) can be used as a discrete electrical model of the power line. For the sake of simplicity, we consider n users who have equal consumption, represented by n equal vertical impedances Z_2, placed at equidistant points characterized by equal horizontal impedances Z_1.

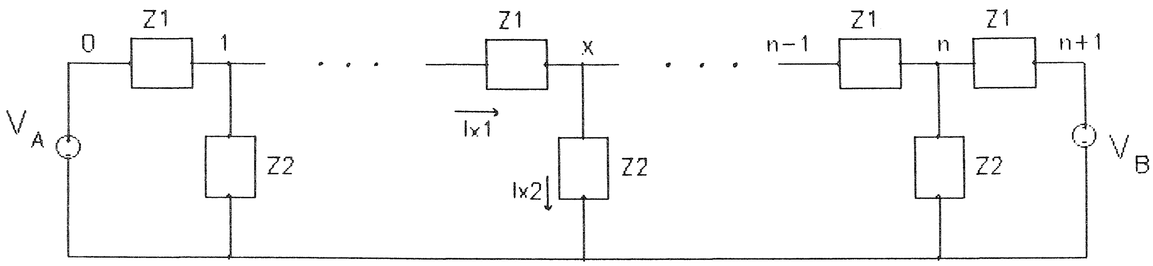

FIGURE 2. Ladder Network as a Model of the Power Line

2. ANALYSIS OF THE LADDER NETWORK

In order to analyze the network of Figure 2, we can use the superimposition of the effects in the networks of Figures 3 and 4. The analysis of these networks can be done starting from the study of the network of Figure 5, by adding a "load" impedance.

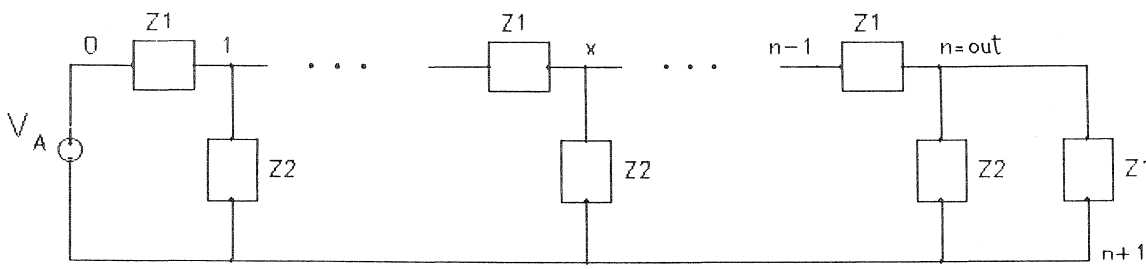

FIGURE 3. Ladder Network Supplied by V_A

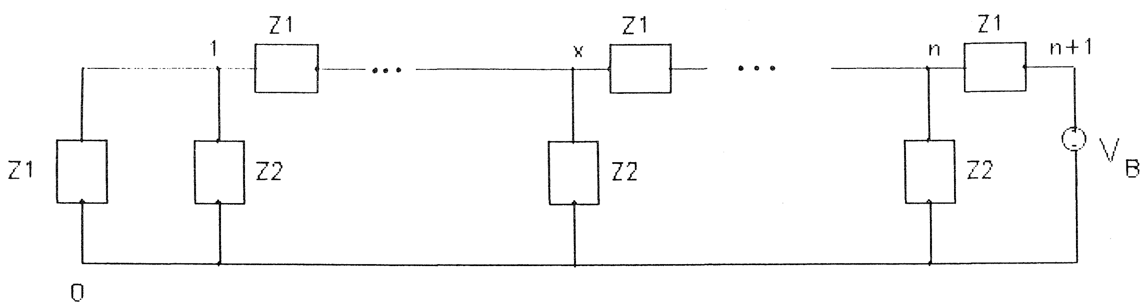

FIGURE 4. Ladder Network Supplied by V_B

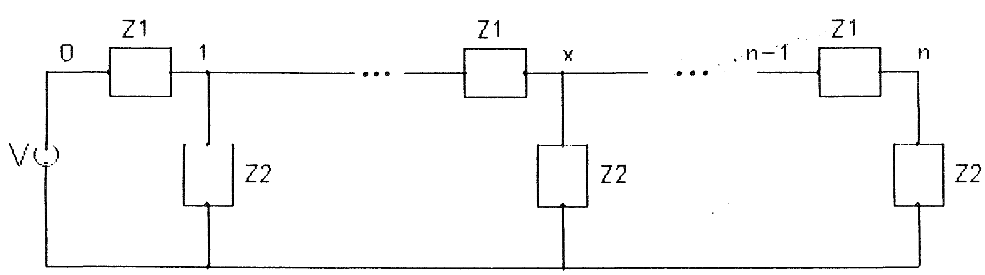

FIGURE 5. Ladder Network with *n* Identical Cells

In [1] a new fast method for the ladder network characterization in Figure 5 was presented; by using this method, all the electrical parameters of a ladder network formed by n identical cells can be written directly by means of both a parameter that characterizes the single cell [the "cell factor" $K(s) = Z_1(s)/Z_2(s)$] and the polynomials in K whose coefficients are the entries of two numerical triangles, named DFF [3] and DFFz [4], here reported:

n	K^0	K^1	K^2	K^3
0	1			
1	1	1		
2	1	3	1	
3	1	6	5	1
	...			

DFF Triangle

$$\text{Entry} = \binom{n+K}{n-K}$$

n	K^0	K^1	K^2	K^3
0	1			
1	2	1		
2	3	4	1	
3	4	10	6	1
	...			

DFFz Triangle

$$\text{Entry} = \binom{n+K+1}{n-K}$$

The mathematical properties of triangles and polynomials have been presented in [2]. Let us call b_n and B_n the polynomials whose coefficients are the entries of DFF and DFFz triangles, respectively. These polynomials coincide with the polynomials defined by Morgan-Voyce and then investigated by Swamy [7] and Lahr [5] and [6].

All the electrical characteristics of the network represented in Figure 5 can be expressed directly in a closed form by means of these polynomials if all the cells are equal.

The networks drawn in Figures 3 and 4 are very similar to that of Figure 5. The only difference is in the fact that the last cell of the Figure 5 network has a "load" impedance of infinite value. It is possible to write the electrical expressions for the Figure 3 and Figure 4 networks as simply as for the Figure 5 ones and also in closed form.

For the Figure 3 network, we have (see [5], p. 275) that the transfer function is given by

$$G'_n(K) = \frac{V_{out}}{V_A} = \frac{1}{B_n(K)}, \tag{1}$$

while the voltage at the generical x^{th} node is given by

$$V'_x(K) = V_A \frac{B_{n-x}(K)}{B_n(K)} \quad (0 \le x \le n+1), \tag{2}$$

with $B_{-1}(K) = 0$.

The voltage behavior for the network of Figure 4 is symmetrical. For that reason, we can write

$$V''_x(K) = V_B \frac{B_{x-1}(K)}{B_n(K)} \quad (0 \le x \le n+1). \tag{3}$$

By the application of the superimposition of the effects, we can write, for the network represented in Figure 2, the following expression for the node voltages:

$$V_x(K) = V'_x(K) + V''_x(K) = V_A \frac{B_{n-x}(K)}{B_n(K)} + V_B \frac{B_{x-1}(K)}{B_n(K)} \quad (0 \le x \le n+1). \tag{4}$$

Denoting by I_{x1} and I_{x2} the currents flowing into the x^{th} cell horizontal and vertical impedances, respectively, we can write similar expressions, using the following property of Morgan-Voyce polynomials, $b_x = B_x - B_{x-1}$ (see [1], [5]-[7]):

$$I_{x1} = \frac{1}{Z_1}[V_x - V_{x-1}] = \frac{1}{Z_1}\left[-V_A \frac{b_{n-x+1}(K)}{B_n(K)} + V_B \frac{b_{x-1}(K)}{B_n(K)}\right] \quad (1 \le x \le n+1); \tag{5}$$

$$I_{x2} = \frac{V_x}{Z_2} = \frac{1}{Z_2}\left[V_A \frac{B_{n-x}(K)}{B_n(K)} + V_B \frac{B_{x-1}(K)}{B_n(K)}\right] \quad (1 \le x \le n). \tag{6}$$

Let us now consider the case of odd n, for which the middle point exists for the voltage and the vertical current and is defined for $x = m = (n+1)/2$. In this point, from (4), we can write

$$V_m = (V_A + V_B)\frac{B_{(n-1)/2}}{B_n}. \tag{7}$$

In the middle vertical impedance, we also have

$$I_{m2} = \frac{1}{Z_2}(V_A + V_B)\frac{B_{(n-1)/2}}{B_n}. \tag{8}$$

In the case of even n, we can reason analogously by considering the middle horizontal current, whose value is given by

$$I_{m1} = \frac{1}{Z_1}(-V_A + V_B)\frac{b_{n/2}}{B_n} \tag{9}$$

being $x = m = (n+2)/2$, while expressions (4)-(6) are always valid.

We are mainly interested in determining the power dissipated in the vertical impedances (because only these have a physical meaning), which is given by the voltage-current product:

$$P_{x2} = \frac{1}{Z_2}\left[V_A \frac{B_{n-x}(K)}{B_n(K)} + V_B \frac{B_{x-1}(K)}{B_n(K)}\right]^2 \quad (1 \le x \le n); \tag{10}$$

$$P_{m2} = \frac{1}{Z_2}(V_A + V_B)^2\left[\frac{B_{(n-1)/2}}{B_n}\right]^2 \quad (n \text{ odd}). \tag{11}$$

The Fibonacci and Lucas numbers appear in the case of $K = 1$, which corresponds to $Z_1 = Z_2 = R$. In this case, $B_x = F_{2x+2}$ and $b_x = F_{2x+1}$. Consequently, we have

$$V_x = V_A \frac{F_{2(n+1-x)}}{F_{2(n+1)}} + V_B \frac{F_{2x}}{F_{2(n+1)}} \quad (0 \le x \le n+1), \tag{12}$$

$$V_m = (V_A + V_B)\frac{F_{n+1}}{F_{2n+2}} = (V_A + V_B)\frac{1}{L_{n+1}} \quad (n \text{ odd}), \tag{13}$$

$$I_{x1} = \frac{1}{R}\left[-V_A \frac{F_{2(n+1-x)+1}}{F_{2(n+1)}} + V_B \frac{F_{2x-1}}{F_{2(n+1)}}\right] \quad (1 \le x \le n+1), \tag{14}$$

$$I_{x2} = \frac{1}{R}\left[V_A \frac{F_{2(n+1-x)}}{F_{2(n+1)}} + V_B \frac{F_{2x}}{F_{2(n+1)}}\right] \qquad (1 \le x \le n), \tag{15}$$

$$I_{m1} = \frac{1}{R}(-V_A + V_B)\frac{1}{L_{n+1}} \qquad (n \text{ even}), \tag{16}$$

$$I_{m2} = \frac{1}{R}(V_A + V_B)\frac{1}{L_{n+1}} \qquad (n \text{ odd}), \tag{17}$$

from which:

$$P_{x2} = \frac{1}{R}\left[V_A \frac{F_{2(n+1-x)}}{F_{2(n+1)}} + V_B \frac{F_{2x}}{F_{2(n+1)}}\right]^2 \qquad (1 \le x \le n); \tag{18}$$

$$P_{m2} = \frac{1}{R}(V_A + V_B)^2\left[\frac{1}{L_{n+1}}\right]^2 \qquad (n \text{ odd}). \tag{19}$$

The last two relations show that the power consumption of the users is also a function of the Fibonacci and Lucas numbers.

3. EXAMPLE

Let us consider the power dissipation in the vertical impedances in the case of $n = 3$, shown in Figure 6 below.

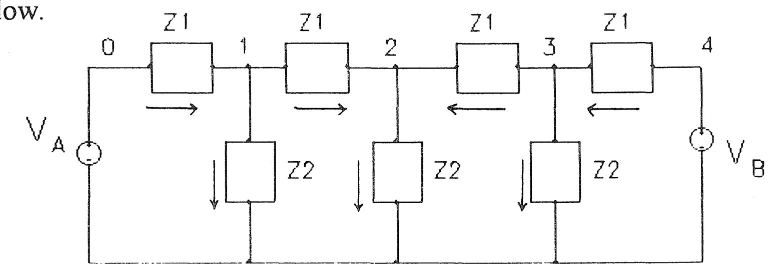

FIGURE 6. Example

In the generical case of different values between the horizontal and vertical impedances, we have, from (9):

$$P_{x2} = \frac{1}{Z_2}\left[V_A \frac{B_{3-x}(K)}{B_3(K)} + V_B \frac{B_{x-1}(K)}{B_3(K)}\right]^2 \quad (1 \le x \le 3), \tag{20}$$

that is,

$$\begin{cases} P_{12} = \dfrac{1}{Z_2}\left[V_A \dfrac{B_2(K)}{B_3(K)} + V_B \dfrac{B_0(K)}{B_3(K)}\right]^2 = \dfrac{1}{Z_2}\left[\dfrac{V_A(K^2+4K+3)+V_B}{K^3+6K^2+10K+4}\right]^2, \\ P_{22} = \dfrac{1}{Z_2}[V_A + V_B]^2\left[\dfrac{B_1(K)}{B_3(K)}\right]^2 = \dfrac{1}{Z_2}[V_A + V_B]^2\left[\dfrac{K+2}{K^3+6K^2+10K+4}\right]^2, \\ P_{32} = \dfrac{1}{Z_2}\left[V_B \dfrac{B_2(K)}{B_3(K)} + V_A \dfrac{B_0(K)}{B_3(K)}\right]^2 = \dfrac{1}{Z_2}\left[\dfrac{V_B(K^2+4K+3)+V_A}{K^3+6K^2+10K+4}\right]^2. \end{cases} \tag{21}$$

In the particular case of $Z_1 = Z_2 = R$, we have

$$P_{x2} = \frac{1}{R}\left[V_A \frac{F_{8-2x}(K)}{F_8(K)} + V_B \frac{F_{2x}(K)}{F_8(K)} \right]^2 \quad (1 \le x \le 3), \tag{22}$$

from which:

$$\begin{cases} P_{12} = [8V_A + V_B]^2 / 441\,R, \\ P_{22} = [V_A + V_B]^2 / 49\,R, \\ P_{32} = [V_B + 8V_A]^2 / 441\,R. \end{cases} \tag{23}$$

4. PARTICULAR SUPPLY VALUES

In the analysis of the symmetrical ladder network, which models the power electrical line, we can consider some particular cases for the values of V_B and V_A.

1) If $V_B = V_A > 0$, the network is completely symmetrical and the current flows as in the direction, for example, indicated in Figure 6, if n is odd. When n is even, in the middle horizontal impedance, the current is zero.

2) If $V_A = -V_B$, and only from the mathematical point of view, only the case n odd is interesting. In this case, in the middle point, all the electrical characteristics (voltage, vertical current and power) are zero.

3) In the case $V_B = V_A + \Delta V$, where ΔV can be positive or negative and $\Delta V \ll V_A, V_B$, we have a slightly unbalanced situation and, as a consequence, there is a small difference in the electrical parameter values. This is a real case and the computation can be of practical importance: if one of the supplies does not have enough power (owing to a lack of power), the other one can provide it. We can write:

$$V_x = V_A \frac{B_{n-x} + B_{x-1}}{B_n} + \Delta V \frac{B_{x-1}}{B_n}; \;\; \Delta V_x = \Delta V \frac{B_{x-1}}{B_n} \quad (1 \le x \le n) \tag{24}$$

and

$$I_{x2} = \frac{1}{Z_2}\left[V_A \frac{B_{n-x} + B_{x-1}}{B_n} + \Delta V \frac{B_{x-1}}{B_n} \right]; \;\; \Delta I_{x2} = \frac{1}{Z_2} \Delta V \frac{B_{x-1}}{B_n} \quad (1 \le x \le n), \tag{25}$$

so that

$$\Delta P_{x2} = \Delta V_x \Delta I_{x2} = \frac{1}{Z_2} \Delta V^2 \left[\frac{B_{x-1}}{B_n} \right]^2 \quad (1 \le x \le n) \tag{26}$$

which, in the case of $Z_1 = Z_2 = R$, is equal to

$$\Delta P_{x2} = \frac{1}{R} \Delta V^2 \left[\frac{F_{2x}}{F_{2n+2}} \right]^2 \tag{27}$$

and, in the middle point, for n odd, is equal to

$$\Delta P_m = \frac{1}{R} \Delta V^2 \left[\frac{1}{L_{n+1}} \right]^2. \tag{28}$$

This means that the power variation is strongly dependent on the number of cells n (i.e., the number of the users) upon whom the line is modeled and is also a function of Fibonacci and Lucas numbers.

For example, if $n = 3$, for a variation of 1%, we have that

$$\mathrm{R} \cdot \Delta \mathrm{P_m} = 2.041 \mu\mathrm{W} \cdot \Omega \tag{29}$$

while, for a variation of 10%, we have that

$$\mathrm{R} \cdot \Delta \mathrm{P_m} = 0.204 \mathrm{mW} \cdot \Omega, \tag{30}$$

where, in the case of 10 cells, we have, for $\Delta \mathrm{V} = 1\%$,

$$\mathrm{R} \cdot \Delta \mathrm{P_m} = 2.52 \mathrm{nW} \cdot \Omega, \tag{31}$$

and, for $\Delta \mathrm{V} = 10\%$,

$$\mathrm{R} \cdot \Delta \mathrm{P_m} = 0.25 \mu\mathrm{W} \cdot \Omega. \tag{32}$$

CONCLUSION

A symmetrical ladder network with a high number of cells can be considered as a good model for the investigation of the behavior of an electrical power line. In the particular case of equal impedances, the electrical characteristics can be written as a function of Fibonacci and Lucas numbers.

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