A NOTE ON THE BRACKET FUNCTION TRANSFORM

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(*Submitted November 1995*)

Let (a_n) be a given sequence. The bracket function transform (s_n) is defined by

$$s_n = \sum_{k=1}^{n} \left[\frac{n}{k}\right] a_k. \tag{1}$$

Let $S(x)$ denote the formal power series of the sequence (s_n), that is,

$$S(x) = \sum_{n=1}^{\infty} s_n x^n.$$

H. W. Gould [2] pointed out that

$$S(x) = \frac{1}{1-x} \sum_{n=1}^{\infty} a_n \frac{x^n}{1-x^n}. \tag{2}$$

The aim of this paper is to study the effect of the terms $\frac{1}{1-x}$, $\frac{1}{1-x^n}$, and x^n in (2). We replace these terms with the powers $\frac{1}{(1-x)^r}$, $\frac{1}{(1-x^n)^s}$, and x^{tn} and find the coefficients of the modified series.

First, we study the effect of the term $\frac{1}{1-x}$. If the term $\frac{1}{1-x}$ is deleted from (2), that is, if

$$T(x) = \sum_{n=1}^{\infty} a_n \frac{x^n}{1-x^n}, \tag{3}$$

then $T(x) = (1-x)S(x)$ and, consequently,

$$t_n = s_n - s_{n-1} = \sum_{k=1}^{n} \left(\left[\frac{n}{k}\right] - \left[\frac{n-1}{k}\right]\right) a_k = \sum_{d \mid n} a_d \tag{4}$$

(see [2], Eq. (8)). More generally, let

$$T(x) = \frac{1}{(1-x)^r} \sum_{n=1}^{\infty} a_n \frac{x^n}{1-x^n}, \quad r \in \mathbf{R}. \tag{5}$$

What are the coefficients of $T(x)$?

Let $\binom{a}{n} = \frac{a(a-1)\cdots(a-n+1)}{n!}$, $a \in \mathbf{R}$. Then

$$\sum_{n=0}^{\infty} \binom{a}{n} x^n = (1+x)^a \tag{6}$$

(see [1], Eq. (1.1)). Thus,

$$\frac{1}{(1-x)^{r-1}} = \sum_{n=0}^{\infty} (-1)^n \binom{-r+1}{n} x^n. \tag{7}$$

It is known (see [2], Eq. (5)) that

$$\frac{1}{(1-x)(1-x^k)}=\sum_{n=0}^{\infty}\left[\frac{n+k}{k}\right]x^n=\sum_{n=0}^{\infty}([n/k]+1)x^n. \tag{8}$$

Combining (7) and (8) and applying the Cauchy convolution, we obtain

$$\frac{1}{(1-x)^r(1-x^k)}=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}(-1)^i\binom{-r+1}{i}\left[\frac{n-i+k}{k}\right]\right)x^n. \tag{9}$$

For the sake of brevity, we write

$$C(n,k,r)=\sum_{i=0}^{n}(-1)^i\binom{-r+1}{i}\left[\frac{n-i+k}{k}\right]. \tag{10}$$

Now we use (9) in finding the coefficients of $T(x)$ in (5). In fact,

$$\begin{aligned}T(x)&=\sum_{k=1}^{\infty}a_kx^k\frac{1}{(1-x)^r(1-x^k)}=\sum_{k=1}^{\infty}a_kx^k\sum_{n=0}^{\infty}C(n,k,r)x^n\\&=\sum_{k=1}^{\infty}a_kx^k\sum_{n=k}^{\infty}C(n-k,k,r)x^{n-k}=\sum_{n=1}^{\infty}x^n\sum_{k=1}^{n}C(n-k,k,r)a_k,\end{aligned}$$

which shows that the coefficients of $T(x)$ in (5) are

$$t_n=\sum_{k=1}^{n}C(n-k,k,r)a_k, \tag{11}$$

where $C(n-k,k,r)$ is as defined in (10). Note that

(i) if $r=1$, then $C(n-k,k,r)=[n/k]$, and thus $t_n=s_n$, which is the bracket function transform (1),

(ii) if $r=0$, then $C(n-k,k,r)=[n/k]-[(n-1)/k]$, and thus (11) reduces to (4).

Second, we study the effect of the term $\frac{1}{1-x^n}$. If the term $\frac{1}{1-x^n}$ is deleted from (2), that is, if

$$T(x)=\frac{1}{1-x}\sum_{n=1}^{\infty}a_nx^n, \tag{12}$$

then

$$t_n=\sum_{k=1}^{n}a_k. \tag{13}$$

More generally, let

$$T(x)=\frac{1}{1-x}\sum_{n=1}^{\infty}a_n\frac{x^n}{(1-x^n)^s},\quad s\in\mathbf{R}. \tag{14}$$

What are the coefficients of $T(x)$?

By (6) we obtain

$$\frac{1}{(1-x)(1-x^k)^s}=(1+x+x^2+\cdots)\left(1-\binom{-s}{1}x^k+\binom{-s}{2}x^{2k}-\cdots\right)=$$

$$= (1+x+\cdots+x^{k-1}) + \left(1-\binom{-s}{1}\right)(x^k + x^{k+1} + \cdots + x^{2k-1})$$
$$+ \left(1-\binom{-s}{1}+\binom{-s}{2}\right)(x^{2k} + x^{2k+1} + \cdots + x^{3k-1}) + \cdots$$
$$= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{[n/k]}(-1)^i\binom{-s}{i}\right)x^n.$$

Applying Equation (1.9) of [1], we obtain

$$\frac{1}{(1-x)(1-x^k)^s} = \sum_{n=0}^{\infty}\binom{[n/k]+s}{[n/k]}x^n. \tag{15}$$

We can use this formula in finding the coefficients of $T(x)$. In fact,

$$T(x) = \sum_{k=1}^{\infty} a_k x^k \frac{1}{(1-x)(1-x^k)^s} = \sum_{k=1}^{\infty} a_k x^k \sum_{n=0}^{\infty}\binom{[n/k]+s}{[n/k]}x^n$$
$$= \sum_{k=1}^{\infty} a_k x^k \sum_{n=k}^{\infty}\binom{[n/k]+s-1}{[n/k]-1}x^{n-k} = \sum_{n=1}^{\infty} x^n \sum_{k=1}^{n}\binom{[n/k]+s-1}{[n/k]-1}a_k,$$

which shows that the coefficients of $T(x)$ in (14) are

$$t_n = \sum_{k=1}^{n}\binom{[n/k]+s-1}{[n/k]-1}a_k. \tag{16}$$

Note that

(i) if $s=1$, then $\binom{[n/k]+s-1}{[n/k]-1} = [n/k]$, and thus $t_n = s_n$, which is the bracket function transform (1),

(ii) if $s=0$, then $\binom{[n/k]+s-1}{[n/k]-1} = 1$, and thus (16) reduces to (13).

Third, we study the effect of the term x^n. Let

$$T(x) = \frac{1}{1-x}\sum_{n=1}^{\infty} a_n \frac{x^{tn}}{1-x^n}, \quad t \in \mathbf{Z}^+. \tag{17}$$

Then, by (8),

$$\begin{aligned} T(x) &= \sum_{k=1}^{\infty} a_k x^{tk} \frac{1}{(1-x)(1-x^k)} = \sum_{k=1}^{\infty} a_k x^{tk} \sum_{n=0}^{\infty}([n/k]+1)x^n \\ &= \sum_{k=1}^{\infty} a_k x^{tk} \sum_{n=tk}^{\infty}\big([(n-tk)/k]+1\big)x^{n-tk} = \sum_{n=t}^{\infty} x^n \sum_{k=1}^{[n/t]}\big([n/k]-t+1\big)a_k, \end{aligned} \tag{18}$$

which shows that the coefficients of $T(x)$ in (17) are

$$t_n = \sum_{k=1}^{[n/t]}\big([n/k]-t+1\big)a_k. \tag{19}$$

Note that if $t=1$, then $t_n = s_n$, which is the bracket function transform (1).

What is the effect of deleting the term x^n in (2), that is, what are the coefficients of

$$T(x)=\frac{1}{1-x}\sum_{n=1}^{\infty}a_n\frac{1}{1-x^n}? \tag{20}$$

Proceeding in a way similar to that in (18), we obtain the coefficients of $T(x)$ in (20) as

$$t_n=\sum_{k=1}^{\infty}([n/k]+1)a_k=s_n+a, \tag{21}$$

provided that the series $\sum_{k=1}^{\infty}a_k$ is convergent and its sum is equal to a.

Finally, we note that the three cases (5), (14), and (17) could be treated simultaneously. In fact, let

$$T(x)=\frac{1}{(1-x)^r}\sum_{n=1}^{\infty}a_n\frac{x^{tn}}{(1-x^n)^s},\quad r,s\in\mathbf{R}, t\in\mathbf{Z}^+. \tag{22}$$

Then

$$t_n=\sum_{k=1}^{[n/t]}C(n-tk,k,r,s)a_k, \tag{23}$$

where

$$C(n,k,r,s)=\sum_{i=0}^{n}(-1)^i\binom{-r+1}{i}\binom{[(n-i)/k]+s}{[(n-i)/k]}.$$

This can be proved in a similar way to the above three cases. For the sake of brevity, we omit the details here.

REFERENCES

1. H. W. Gould. *Combinatorial Identities.* Printed by Morgantown Printing and Binding Co., 1972.
2. H. W. Gould. "A Bracket Function Transform and Its Inverse." *The Fibonacci Quarterly* **32.2** (1994):176-79.

AMS Classification Numbers: 05A15, 05A19, 11B83