SOME PROPERTIES OF THE GENERALIZED FIBONACCI SEQUENCES $C_n = C_{n-1} + C_{n-2} + r$

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The generalized Fibonacci sequences $\{C_n(a,b,r)\}$ defined by $C_n(a,b,r) = C_{n-1}(a,b,r) + C_{n-2}(a,b,r) + r$ with $C_1(a,b,r) = a$, $C_2(a,b,r) = b$, where r is a constant, have been studied in [2] and [3]. Again we take the initial value $C_0(a,b,r) = b-a-r$. The Fibonacci sequence arises as a special case, $F_n = C_n(1,1,0)$, while the Lucas sequence is $L_n = C_n(1,3,0)$.

The purpose of this note is to establish some properties of $C_n(a,b,r)$ by using the method of L. C. Hsu [1].

For the convenience of the reader, we introduce the following symbols:

I will be the identity operator;

E represents the shift operator;

E_i is the "i^{th} coordinate" shift operator ($i = 1, 2$);

$\nabla = I + E_2 - E_1$.

We also let $\binom{n}{i,j} = \dfrac{n!}{i!\,j!\,(n-i-j)!}$.

In [1], Hsu and Maosen gave the following proposition.

Proposition 1: Let $f(n,k)$ and $g(n,k)$ be any two sequences. Then the following reciprocal formulas hold:

$$g(n,k) = \nabla^n f(0,k) = \sum_{i+j+s=n} \binom{n}{i,j} (-1)^i f(i, k+j), \tag{1}$$

$$f(n,k) = \nabla^n g(0,k) = \sum_{i+j+s=n} \binom{n}{i,j} (-1)^i g(i, k+j). \tag{2}$$

From this point on, we briefly write C_n for $C_n(a,b,r)$.

Lemma 1: $C_k + C_{k+1} + C_{k+6} = 3C_{k+4}$. (3)

Proof:

$$\begin{aligned} C_k + C_{k+1} + C_{k+6} &= C_k + C_{k+1} + C_{k+5} + C_{k+4} + r \\ &= C_{k+2} - r + C_{k+4} + C_{k+3} + r + C_{k+4} + r \\ &= C_{k+4} - r + C_{k+4} + r + C_{k+4} = 3C_{k+4.} \end{aligned}$$

Theorem 1:

$$C_{4n+6k} = \sum_{i+j+s=n} \binom{n}{i,j} 3^{-n} C_{i+6(j+k)}, \tag{4}$$

$$C_{n+6k} = \sum_{i+j+s=n} \binom{n}{i,j} (-1)^{n+i} 3^i C_{4i+6(j+k)}. \tag{5}$$

Proof: We take $f(i,j) = (-1)^i C_{i+6j}$. Using Lemma 1,

$$\begin{aligned}\nabla f(i, j) &= (I + E_2 - E_1) f(i, j) = f(i, j) + f(i, j+1) - f(i+1, j)\\ &= (-1)^i C_{i+6j} + (-1)^i C_{i+6(j+1)} - (-1)^{i+1} C_{i+1+6j}\\ &= (-1)^i (C_{i+6j} + C_{i+6j+1} + C_{i+6j+6}) = (-1)^i 3C_{i+6j+4}.\end{aligned}$$

Hence, $\nabla \equiv 3E_1^4$. Thus, we obtain $g(n, k) = \nabla^n f(0, k) = 3^n E_i^{4n} f(0, k) = 3^n C_{4n+6k}$. By (1), we have

$$3^n C_{4n+6k} = \sum_{i+j+s=n} \binom{n}{i,\, j} C_{i+6(j+k)},$$

and, by (2), we get

$$(-1)^n C_{n+6k} = \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^i 3^i C_{4i+6(j+k)},$$

completing the proof of Theorem 1.

We take $k = 0$ in Theorem 1 to Write Corollary 1.1, and $i = 0$ in Corollary 1.1 to derive Corollary 1.2.

Corollary 1.1:
$$C_{4n} = \sum_{i+j+s=n} \binom{n}{i,\, j} 3^{-n} C_{i+6j}, \tag{6}$$
$$C_n = \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^{n+i} 3^i C_{4i+6j}. \tag{7}$$

Corollary 1.2:
$$C_n - (-1)^n \sum_{j=0}^{n} \binom{n}{j} C_{6j} \equiv 0 \pmod 3. \tag{8}$$

We can obtain Theorem 2, in a manner similar to that used to prove Theorem 1, by taking $f(i, j) = (-1)^i C_{6i+j}$ and expanding $\nabla f(i, j)$. Again, set $k = 0$ in Theorem 2 to write Corollary 2.1, and let $i = 0$ in (12) below to obtain Corollary 2.2.

Theorem 2:
$$C_{4n+k} = \sum_{i+j+s=n} \binom{n}{i,\, j} 3^{-n} C_{6i+j+k}, \tag{9}$$
$$C_{6n+k} = \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^{n+i} 3^i C_{4i+j+k}. \tag{10}$$

Corollary 2.1:
$$C_{4n} = \sum_{i+j+s=n} \binom{n}{i,\, j} 3^{-n} C_{6i+j} \tag{11}$$
$$C_{6n} = \sum_{i+j+s=n} \binom{n}{i,\, j} (-1)^{n+i} 3^i C_{4i+j}. \tag{12}$$

Corollary 2.2:
$$C_{6n} - (-1)^n \sum_{j=0}^{n} \binom{n}{j} C_j \equiv 0 \pmod 3. \tag{13}$$

Proposition 2: If a sequence $\{X_n\}$ satisfies

$$I = 2E^{-1} - E^{-3} \tag{14}$$

then

$$I = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} 2^i E^{-3n+2i}; \tag{15}$$

hence,

$$X_{3n} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} 2^i X_{2i}, \tag{16}$$

and

$$X_{3n+k} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} 2^i X_{2i+k}. \tag{17}$$

Proof: Use binomial expansions.

Lemma 2: $C_n = 2C_{n-1} - C_{n-3}$. (18)

Proof:

$$\begin{aligned} C_n &= C_{n-1} + C_{n-2} + r \\ &= C_{n-1} + C_{n-1} - C_{n-3} - r + r \\ &= 2C_{n-1} - C_{n-3}. \end{aligned}$$

Theorem 3:

$$C_{3n} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} 2^i C_{2i}, \tag{19}$$

$$C_{3n+k} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} 2^i C_{2i+k}. \tag{20}$$

Proof: Since C_n satisfies (14), Theorem 3 is proved by Proposition 2.

Our final corollary follows by setting $i = 0$ in (20).

Corollary 3.1: $C_{3n+k} - (-1)^n C_k \equiv 0 \pmod 2$. (21)

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