PROFESSOR LUCAS VISITS THE PUTNAM EXAMINATION

Leon C. Woodson
Department of Mathematics, Morgan State University, Baltimore, MD 21239-4001
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The following problem appeared on the 1995 William Lowell Putnam Mathematical Competition:

Evaluate

$$\sqrt[8]{2207-\cfrac{1}{2207-\cfrac{1}{2207-\cdots}}}$$

and express the answer in the form $(a+b\sqrt{c})/d$, where a, b, c, and d are integers.

Readers of this journal might recognize that 2207 is the sixteenth Lucas number, L_{16}. Therefore, a more general problem is to evaluate

$$\sqrt[n]{L_{2n}-\cfrac{1}{L_{2n}-\cfrac{1}{L_{2n}-\cdots}}}\ .$$

Let S denote this expression. Then $S^n = L_{2n}-(1/S^n)$, and therefore $S^{2n}-L_{2n}S^n+1=0$. It follows that

$$S^n = \frac{L_{2n}+\sqrt{L_{2n}^2-4}}{2}.$$

Now, using the Binet formula for the Lucas numbers, i.e., $L_n - \alpha^n + \beta^n$, $\alpha=(1+\sqrt{5})/2$, and $\beta=(1+\sqrt{5})/2$, we have

$$S^n = \frac{\alpha^{2n}+\beta^{2n}+\sqrt{(\alpha^{2n}+\beta^{2n})^2-4}}{2} = \frac{\alpha^{2n}+\beta^{2n}+\sqrt{(\alpha^{2n}-\beta^{2n})^2}}{2} = \alpha^{2n}.$$

It follows that $S=\alpha^2=(3+\sqrt{5})/2$ (independent of n).

This technique can be used to simplify a variety of expressions of this general form. A more natural solution to the original problem is to set T, say, equal to the expression and note, as above, that

$$T^{16}-2207T^8+1=0.$$

This can be written in the form $T^{16}+2T^8+1=47^2T^8$. Then, taking square roots (T is positive), we have $T^8+1=47T$. Repeating this gives $T^4+1=7T^2$ and $T^2+1=3T$. Solving the latter equation yields the solution $T=(3+\sqrt{5})/2$.

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