

# THE PASCAL-DE MOIVRE TRIANGLES\*

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## 1. INTRODUCTION

The coefficients of the Pascal triangle were generalized in 1756 by de Moivre [5]. Each row of a Pascal triangle contains a sequence of numbers that are the coefficients of the power series expansion for the binary expression  $(1+x)^N$ . The de Moivre formula [2], [4], [5], [6] derives the coefficients of the power series for the generalized expansion of  $(1+x+x^2+\dots+x^{(J-1)})^N$ . Thus, for integers  $(J \geq 2$  and  $N \geq 1)$  and for  $0 \leq h \leq N(J-1)$ , we define  $C(N, J; h)$  to be the coefficients of  $(x^h)$  in the expansion of

$$(1+x+x^2+\dots+x^{(J-1)})^N = \sum C(N, J; h)x^h. \quad (1)$$

A Pascal-de Moivre triangle can be created from the coefficients  $C(N, J; h)$  for each positive integer value  $(J)$ . For example, with  $(J=3)$ , the Pascal-de Moivre triangle of  $C(N, J; h)$  terms for row numbers  $1 \leq N \leq 4$  is:

$N \setminus h$	0	1	2	3	4	5	6	7	8
1	1	1	1						
2	1	2	3	2	1				
3	1	3	6	7	6	3	1		
4	1	4	10	16	19	16	10	4	1

(2)

In this paper, the sequence of  $C(N, J; h)$  terms in each row  $(N)$  of the Pascal-de Moivre triangle is examined for the series properties, at various arrangements of terms. Each  $C(N, J; h)$  term in the  $N^{\text{th}}$  row of the Pascal-de Moivre triangle is assigned a coefficient factor  $(F_h)$ , such that

$$\text{Series with coefficients } (F) = \sum \{(F_h) C(N, J; h)\}. \quad (3)$$

Sections 3 and 4 define summations of all  $C(N, J; h)$  terms in the  $N^{\text{th}}$  row of the Pascal-de Moivre triangle that are separated by some fixed interval spacing  $(\Delta h)$ . Then, from the set of coefficients  $(F)$ , the factors  $(F_h)$  equal one at each interval step and equal zero otherwise. Section 3 examines these summations of the  $C(N, J; h)$  terms at intervals that are a function of the distribution variable  $(J)$ ; i.e., for  $(\Delta h = f(J))$ . The quadruplet cycle of Section 4 adds the  $C(N, J; h)$  terms with interval spacing  $(\Delta h = 4)$ .

In Sections 5 and 6, the coefficient factors  $(F_h)$  are related to the moments of the  $C(N, J; h)$  distribution. A quick review of the theory of moments from [3] will illustrate which coefficient factors  $(F_h)$  from set  $(F)$  are involved, and what form the series in (3) will take.

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The moment  $(m_{(R,x)})$  about a point  $(x)$  for a discrete distribution of  $(C_h)$  terms can be expressed by a summation over all term indices  $(h)$ . The  $R^{\text{th}}$  moment of  $(m_{(R,x)})$  is defined by the summation in (4) for the distribution density  $f(h)$  evaluated at each index  $(h)$ :

$$m_{(R,x)} = \sum\{(h-x)^R f(h)\}, \tag{4}$$

where  $f(h) = (C_h) / (\sum C_h)$ .

We choose the distribution terms  $(C_h)$  to be the terms in the  $N^{\text{th}}$  row of the Pascal-de Moivre triangle, given by  $C(N, J; h)$ . By rearranging equation (4), we define a moment summation equation (5) for the  $C(N, J; h)$  distribution as:

$$\sum\{(h-x)^R C(N, J; h)\} = (m_{(R,x)}) (\sum\{C(N, J; h)\}). \tag{5}$$

The left-hand side of equation (5) is the same as equation (3) with the coefficient factors  $(F_h) = (h-x)^R$ . Section 5 uses equation (5) to obtain Summations Based on Moments. Section 6 uses a similar equation for (3), but with  $(F_h) = (-1)^h (h-x)^R$ , to obtain Summations Based on Alternating Signed Terms. In Sections 5 and 6, the  $C(N, J; h)$  moment summations and moments  $(m_{(R,x)})$  are evaluated relative to points at  $(x = 0, \text{ the origin})$  and  $(x = M, \text{ the mean})$ .

## 2. DERIVATION AND TERMINOLOGY

De Moivre derived the formula for each  $C(N, J; h)$  term by writing the left-hand side of equation (1) in the form  $(1-x^J)^N (1-x)^{-N}$ , expanded both factors with the binomial theorem, and collected terms. The resulting formula (6) is a summation over all integers  $\{0 \leq a \leq [h/J]\}$ , where  $[h/J]$  is the "least integer function" for the largest integer not exceeding the value of  $h/J$ :

$$C(N, J; h) = \sum C(N, J; h, a) = \sum (-1)^a \binom{h-aJ+N-1}{N-1} \binom{N}{a}. \tag{6}$$

In a reduced format of factorials, with the substitution  $(N) = (N!) / ((N-1)!)$ , the de Moivre formula becomes the summation in (7) over all integers  $\{0 \leq a \leq [h/J]\}$ :

$$C(N, J; h) = \sum (-1)^a \frac{(h-aJ+N-1)!}{(h-aJ)!} \frac{(N)}{(N-a)!(a)!}. \tag{7}$$

A standard terminology will be used for the coefficient terms of the Pascal-de Moivre triangles. A consistent notation for the  $C(N, J; h)$  and  $C(N, J; h, a)$  terms is described here:

- $C$  capital letter for the term itself (the coefficient of the basic expansion);
- $N, J$  capital letters for the independent variables of the  $C(N, J; h)$  series;
- $h, a$  small letters for the summation indices in their respective sums.

The power of the *Mathematica* [8] program allowed computations that could accurately generate numbers in excess of 100 digits. Therefore, large  $C(N, J; h)$  distributions were evaluated with precision, including those defined by  $(N, J)$  values of (100, 2) and (20, 20) and (2, 300).

## 3. COLUMNAL SUMMATIONS

The full  $N^{\text{th}}$  row sequence of terms  $C(N, J; h)$  in the Pascal-de Moivre triangle has a known series value of  $(J^N)$  per [4], [7], when summed over all integers  $0 \leq h \leq N(J-1)$ :

$$\sum C(N, J; h) = (J^N). \tag{8}$$

The  $C(N, J; h)$  sequence can also be partitioned by taking every ( $Q^{\text{th}}$ ) term to obtain an ordered summation  $S(N, J; Q, r)$  of the  $C(N, J; h)$  terms. Reference to such partitioning is given for the binomial ( $J = 2$ ) case in [9] and for the  $C(N, J; h)$  sequence in [1]. Here and in the next section, the derivation of  $S(N, J; Q, r)$  will use a variation of the methods described in those references. The main difference between Hoggatt's approach in [1] and the one employed in this section is that here the least integer function for  $[J^N / Q]$  is used rather than the simple ratio of  $(J^N / Q)$ .

For pictorial convenience, the partition of a  $C(N, J; h)$  sequence can be displayed in tabular form with ( $Q$ ) columns. As a guide to the tabular display of partitions, the  $C(N, J; h)$  sequence at values  $(N, J) = (3, 3)$  from (2) will be analyzed for various spacings ( $Q$ ) to obtain the sums  $S(N, J; Q, r)$ .

**Table of Columnal Sums  $S(N, J; Q, r)$  for  $(N, J) = (3, 3)$**  (9)

	$Q = (J - 1) = 2$		$Q = J = 3$			$Q = (J + 1) = 4$			
<i>A</i>	<i>r</i> = 0	1	0	1	2	0	1	2	3
0	1	3	1	3	6	1	3	6	7
1	6	7	7	6	3	6	3	1	
2	6	3	1						
3	1								
$S(N, J; Q, r)$	14	13	9	9	9	7	6	7	7
	$[J^N / Q] = [3^3 / 2] = 13$		$[J^N / Q] = [3^3 / 3] = 9$			$[J^N / Q] = [3^4 / 4] = 6$			

Each row will have a row number ( $A$ ) with values from  $0 \leq A \leq [N(J-1)/Q]$ , where the brackets indicate the least integer function for the greatest integer not exceeding the enclosed expression. The columns in this table will have column numbers ( $r$ ) in the range  $0 \leq r \leq (Q-1)$ . The values of the column series  $S(N, J; Q, r)$  are analyzed for various interval spacings ( $Q$ ) at or near the value ( $J$ ).

For  $Q = J$ , the terms at  $(h = AQ + r)$  will be summed over integers  $0 \leq A \leq [N(J-1)/(J)]$ . Each column ( $r$ ) in the range  $0 \leq r \leq (J-1)$  will have the sum for  $S(N, J; Q, r)$  per equation (10). Table (9) above shows that the  $C(N, J; h)$  sequence at  $(N, J) = (3, 3)$  has each columnal sum  $S(N, J; Q, r)$  equal to (9).

$$S(N, J; Q, r) = \sum C(N, J; h) = J^{(N-1)} \text{ for each } (r). \tag{10}$$

In the binomial case of ( $J = 2$ ) for Pascal's (classical) triangle, formula (10) generalizes the familiar fact that the sum of alternate terms ( $Q = 2$ ) in any row is half the sum of the entire row from (8), since  $(J^{N-1}) = 2^{(N-1)} = (1/2)2^N = (1/2)(J^N)$  when ( $J = 2$ ). It does not mean, however, that the sum of alternate terms of the generalized Pascal-de Moivre triangle with ( $J \neq 2$ ) is half the sum of the row terms (a case that is dealt with in equations (19) and (20) of the next section).

For  $Q = (J-1)$ , the  $C(N, J; h)$  terms at  $(h = AQ + r)$  are summed over integers  $0 \leq A \leq N$ . So then, for column locations ( $r$ ) in the range  $0 \leq r \leq (J-2)$ , the column sums  $S(N, J; Q, r)$  will

be given by equation (11) for all  $N \geq 1$ . In the  $(N, J) = (3, 3)$  example, the first column at  $(r = 0)$  has a sum of (14), which is one more than the sum of (13) at the other column(s), as seen in table (9). Equation (8) covers the linear  $(Q = 1)$  case at  $(J = 2)$ .

$$S(N, J; Q, r) = \sum C(N, J; h) = [(J^N)/(J-1)] + 1 \quad \text{for } J > 2 \text{ and } r = 0$$

$$= [(J^N)/(J-1)] \quad \text{for } J > 2 \text{ and } 1 \leq r \leq (J-2). \tag{11}$$

For  $Q = (J+1)$ , the column terms at  $(h = AQ+r)$  are summed over the integers  $0 \leq A \leq [N(J-1)/(J+1)]$ . So, for column locations  $(r)$  in the range  $0 \leq r \leq J$ , the sums  $S(N, J; Q, r)$  satisfy equation (12):

$$S(N, J; Q, r) = \sum C(N, J; h) = [(J^N)/(J+1)] + K. \tag{12}$$

The value of  $(K)$  in equation (12) is either one or zero, as determined in table (13) below.

**Table for  $K$  Values** (13)

Column Type	Condition for Column Type	$N = \text{Odd}$	$N = \text{Even}$
Unique	$(N+r) = 0 \pmod{(J+1)}$	$K = 0$	$K = 1$
Common	$(N+r) \neq 0 \pmod{(J+1)}$	$K = 1$	$K = 0$

In these  $Q = (J+1)$  cases, one column is always unique. All the other columns will have identical sums that differ from the unique column by one. In the  $(N, J) = (3, 3)$  example, where  $(N)$  is odd, the table for  $K$  values (13) indicates that the *unique* column will have  $(K = 0)$ . Thus, from equation (12) and table (9), the unique column sum  $S(N, J; Q, r)$  equals (6). At  $N = 3$ , the only column location  $(r)$  within  $0 \leq r \leq 3$  that satisfies  $(N+r) = 0 \pmod{4}$  is at  $(r = 1)$ , per the Condition in table (13). All other columns have  $(K = 1)$  and *common* column sums  $S(N, J; Q, r)$  equaling (7).

So in general, for any interval spacing  $Q = \{J+1, J, \text{ or } J-1\}$ , each columnal series with terms at  $h = (AQ+r)$  is summed over integers  $0 \leq A \leq [N(J-1)/(Q)]$ . For the partition of the  $C(N, J; h)$  sequence with these spacings  $(Q)$ , each column  $(r)$  in the range  $0 \leq r \leq (Q-1)$  will yield a series result for  $S(N, J; Q, r)$  given by equation (14):

$$S(N, J; Q, r) = [(J^N)/(Q)] + b, \text{ where } b = \{0 \text{ or } 1\}. \tag{14}$$

#### 4. SUMMATIONS WITH QUADRUPLET CYCLES

The Method of Ramus described in [1] and [9] uses the roots of unity, with its real and imaginary parts, to partition the terms in the  $N^{\text{th}}$  row of the Pascal and the Pascal-de Moivre triangles. The terms of the  $C(N, J; h)$  sequence are likewise segmented here by using the second and fourth roots of unity. This segmentation creates four equations of series ( $A$  through  $D$ ) in table (15), whose coefficients  $(F_r)$  repeat for every fourth term of the  $C(N, J; h)$  sequence.

These repeating coefficients  $(F_r)$  in table (15) are the same as the coefficients  $(F_h)$  of the  $C(N, J; h)$  terms from equation (3). For any series in (16), each  $(F_h)$  value equals the  $(F_r)$  entry in table (15) when  $\{h = r \pmod{4}\}$ . The sum equation of series ( $A$ ), for example, is expanded in (17).

**Coefficient Table for Series  $\{A, B, C, D\}$  and Quadruplet Partitions  $(P_r)$**  (15)

Coefficients $(F_r)$ at $r =$	0	1	2	3
Series $A$	+1	0	+1	0
Series $B$	0	+1	0	+1
Series $C$	+1	0	-1	0
Series $D$	0	+1	0	-1
Quadruplet $(P_r) =$	$P_0$	$P_1$	$P_2$	$P_3$

$$\text{Series } (A, B, C, D) = \sum\{(F_h)C(N, J; h)\} \text{ for } 0 \leq h \leq N(J-1). \quad (16)$$

$$\text{Series } (A) = \sum\{C(N, J; h = 4t) + C(N, J; h = 4t + 2)\} \text{ for } 0 \leq t \leq N(J-1)/4. \quad (17)$$

The creation of a quadruplet cycle from table (15) using the series  $\{A, B, C, D\}$  equations requires the identification of the relationships between the series equations and the quadruplet  $\{P_0, P_1, P_2, P_3\}$  equations. In the nomenclature of the  $S(N, J; Q, r)$  partition sums from [1], the quadruplet  $\{P_r\}$  equations will be defined as

$$P_r = S(N, J; 4, r) \text{ since } Q = 4 \text{ and for integers } (r) \text{ within } 0 \leq r \leq 3.$$

The  $C(N, J; h)$  sequences thus created, whose sums are  $(P_r)$ , will have spacings between the nonzero terms of  $(\Delta h = 4)$ , compared with the nonzero term spacing of  $(\Delta h = 2)$  for those of the series  $\{A, B, C, D\}$ . The corresponding transformation of the equations  $\{A, B, C, D\}$  into the equations  $\{P_r\}$  is given for each quadruplet location  $(r)$  from table (15) by:

$$\begin{aligned} P_0 &= (A + C)/2, & P_1 &= (B + D)/2, \\ P_2 &= (A - C)/2, & P_3 &= (B - D)/2. \end{aligned} \quad (18)$$

Now, in order to state the quadruplet equations, the actual formulas for the segmentation equations  $\{A, B, C, D\}$  must be obtained. The first two equations  $\{A$  and  $B\}$  are the equations for the sum of alternate terms for  $C(N, J; h)$  in the  $N^{\text{th}}$  row of the Pascal-de Moivre triangle. Here we state, from empirical analysis, that the series  $(A)$  starting at  $(h = 0)$  and the series  $(B)$  starting at  $(h = 1)$  have series summation formulas given by equations (19) and (20), where  $(N \geq 1)$  and  $(b) = J \pmod{2}$ :

$$A = ((J^N) + b) / 2 \quad (19)$$

and

$$B = ((J^N) - b) / 2, \quad (20)$$

where  $b = 0$  for  $(J = \text{even})$ , or  $b = 1$  for  $(J = \text{odd})$ .

The two segmentation series  $\{C$  and  $D\}$  have equations that are obtained from the tables (22) and (23), respectively, with values of  $(\pm 1, 0, \text{ or } \pm S)$ , where  $S$  is defined by equation (21):

$$S = ((-1)^{[N/4]})(2^{[N/2]}), \quad (21)$$

where the bracketed expressions in the exponents are least integer functions.

**Table for Series (C)** (22)

Table for Series (C)	$J = 0 \pmod{4}$	$J = 1 \pmod{4}$	$J = 2 \pmod{4}$	$J = 3 \pmod{4}$
All $N \geq 1$	0	1		
$N = 0 \pmod{4}$			$S$	1
$N = 1 \pmod{4}$			$S$	0
$N = 2 \pmod{4}$			0	-1
$N = 3 \pmod{4}$			$-S$	0

**Table for Series (D)** (23)

Table for Series (C)	$J = 0 \pmod{4}$	$J = 1 \pmod{4}$	$J = 2 \pmod{4}$	$J = 3 \pmod{4}$
All $N \geq 1$	0	0		
$N = 0 \pmod{4}$			0	0
$N = 1 \pmod{4}$			$S$	1
$N = 2 \pmod{4}$			$S$	0
$N = 3 \pmod{4}$			$S$	-1

As an example of quadruplet analysis, the  $C(N, J; h)$  sequence from table (2) with  $(N, J) = (4, 3)$  is listed below in the tabular form of Section 3 with  $(Q = 4)$ . The values  $\{A, B, C, D\}$  are calculated from equations (19), (20), and (21) and tables (22) and (23). Then the  $\{P_r\}$  values, obtained from equations (18), give confirmation of their equality with the corresponding column sums.

Column Sums of the  $C(N, J; h)$  Sequence

$r =$	0	1	2	3
	1	4	10	16
	19	16	10	4
	$\frac{1}{21}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$
Sum =	$\frac{1}{21}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$

Series Equations

$$\begin{aligned}
 A &= (3^4 + 1) / 2 = 41 \\
 B &= (3^4 - 1) / 2 = 40 \\
 C &= 1 \\
 D &= 0
 \end{aligned}$$

Quadruplet  $\{P_r\}$  Equations

$$\begin{aligned}
 P_0 &= (A + C) / 2 = (41 + 1) / 2 = 21 \\
 P_1 &= (B + D) / 2 = (40 + 0) / 2 = 20 \\
 P_2 &= (A - C) / 2 = (41 - 1) / 2 = 20 \\
 P_3 &= (B - D) / 2 = (40 - 0) / 2 = 20
 \end{aligned}$$

So, as just illustrated, the four quadruplet series  $\{P_r\}$  may be built from combinations of the series  $\{A, B, C, D\}$ , according to equations (18). Also any desired arrangement of the four  $\{P_r\}$  equations can be combined further with any choice of coefficients  $(G_r)$  represented as numbers, variables, or functions, as shown:

$$\text{Quadruplet Arrangement} = \sum\{(G_r)(P_r)\} \text{ for } (r) \text{ within } 0 \leq r \leq 3.$$

5. SUMMATIONS BASED ON MOMENTS

The definition of moments about a point for the  $C(N, J; h)$  distribution was introduced by equation (4) in Section 1. The two moments considered here are the moment ( $v$ ) taken about the origin and the moment ( $\mu$ ) taken about the mean ( $M$ ). The connection between  $R^{\text{th}}$  moment calculations and the summations of  $C(N, J; h)$  terms is indicated by their respective evaluation formulas from equation (5), when summed over  $0 \leq h \leq N(J-1)$ .

$$\sum\{(h)^R C(N, J; h)\} = (v_R)(J^N), \tag{24}$$

$$\sum\{(h-M)^R C(N, J; h)\} = (\mu_R)(J^N). \tag{25}$$

The mean ( $M$ ) is the midpoint of the range of ( $h$ ) values and thus equals  $N(J-1)/2$ , which may include half integers when  $N$  and  $(J-1)$  are both odd integers.

The sum of all  $C(N, J; h)$  row terms in a Pascal-de Moivre triangle equals  $(J^N)$ , which was substituted for  $\sum C(N, J; h)$  in the typical definition of moment equations [3]. Multiplied by  $(J^N)$ , the moment equations ( $v_R$ ) and ( $\mu_R$ ) give the  $C(N, J; h)$  moment summations (24) and (25).

To find the moment equations for ( $v_R$ ) and ( $\mu_R$ ), we derive their exponential generating functions: ( $\mu_{\text{egf}}$ ) and ( $v_{\text{egf}}$ ). And then, by expansion of the exponential generating functions, the coefficient of the term  $((t^R)/R!)$  is the equation for the  $R^{\text{th}}$  moments for ( $\mu_R$ ) and ( $v_R$ ), in the summation over integers  $0 \leq i < \infty$ , as outlined in [3]:

$$\mu_{\text{egf}} = \sum(\mu_i)(t^i)/(i!),$$

$$v_{\text{egf}} = \sum(v_i)(t^i)/(i!).$$

The exponential generating function for ( $\mu_{\text{egf}}$ ) turns out to be the exponential power series shown in (26), which is summed over integers  $2 \leq r < \infty$ . The notation  $\text{Exp}\{x\}$  is defined as  $(e^x)$ .

$$\mu_{\text{egf}} = \text{Exp}\{N \sum((-1)^r (S_r)(t^r)/(r!))\}, \tag{26}$$

where  $S_r = ((J^r) - 1)(B_r)/r$  with  $B_r$  = the  $r^{\text{th}}$  Bernoulli number.

Each Bernoulli number ( $B_r$ ) can be derived as the coefficient of  $((t^r)/r!)$  in the exponential generating function ( $B_{\text{egf}}$ ) from the summation in (27) for  $0 \leq i < \infty$ . From reference [10], the sequence of Bernoulli numbers ( $B_r$ ) for  $(r \geq 1)$  is  $\{(-1/2), (1/6), 0, (-1/30), \dots\}$ .

$$B_{\text{egf}} = \sum(B_i)(t^i)/(i!) = t / \{(e^t) - 1\}. \tag{27}$$

The accuracy of the ( $\mu_{\text{egf}}$ ) formula in (26) has been confirmed empirically by comparing the distribution results on the left-hand side of equation (25) with the moment equation results of the right-hand side of equation (25). These two approaches gave identical results through  $R \leq 32$ , which represented the limit at which the author's computer hardware capability could complete the calculations in a reasonable time.

A couple of cases will illustrate the creation of the initial ( $\mu_R$ ) equations, in terms of their distribution variables ( $N$  and  $J$ ). Also, a specific example with  $(N, J) = (2, 3)$  can demonstrate the numerical equality between the left-hand and right-hand sides of equation (25).

For the case in which  $R = 2$ , the coefficient of  $(t^2)/(2!)$  in the expansion of the ( $\mu_{\text{egf}}$ ) formula in (26) gives the variance ( $\mu_2$ ) of the  $C(N, J; h)$  distribution:

$$\begin{aligned} \mu_2 &= N(S_2) = N(J^2 - 1)B_2 / 2, \quad \text{where } B_2 = (1/6), \\ \mu_2 &= N(J^2 - 1) / 12. \end{aligned} \tag{28}$$

When multiplied by  $(J^N)$ , this second moment ( $\mu_2$ ) formula (28) yields the  $C(N, J; h)$  moment summation formula from (25).

$$\sum\{(h - M)^2 C(N, J; h)\} = \mu_2(J^N) = N(J^2 - 1)(J^N) / 12. \tag{29}$$

The result of (29) generalizes the fact that the second moment summation of terms in the  $N^{\text{th}}$  row of Pascal's (classical) triangle equals the value  $N(2^{N-2})$ , as shown in (30) for  $(J = 2)$ :

$$\sum\{(h - M)^2 C(N, 2; h)\} = \mu_2(2^N) = N(2^2 - 1)(2^N) / 12 = N(2^{N-2}). \tag{30}$$

For symmetrical distributions like  $C(N, J; h)$ , the moment ( $\mu_R$ ) about the mean ( $M$ ) will be zero whenever  $R$  is an odd integer. Thus, the coefficients of  $(t^{2i+1}) / ((2i + 1)!) in the expansion of ( $\mu_{\text{egf}}$ ) must be zero, for all integers ( $i \geq 1$ ). All of these coefficients ( $\mu_{2i+1}$ ) contain a factor with a Bernoulli number of odd index, which is zero per [10]. So the moments ( $\mu_R$ ) are zero when  $R$  is an odd integer, because the corresponding Bernoulli numbers are zero.$

For the case in which  $R = 4$ , the coefficient of  $(t^4) / (4!)$  in the expansion of formula (26) for ( $\mu_{\text{egf}}$ ) gives the 4<sup>th</sup> moment ( $\mu_4$ ):

$$\begin{aligned} \mu_4 &= \{3N^2(S_2)^2\} + \{N(S_4)\}, \\ \mu_4 &= \{3N^2((J^2 - 1)B_2 / 2)^2\} + \{N((J^4 - 1)B_4 / 4)\}, \quad \text{where } B_4 = (-1/30), \\ \mu_4 &= \{N^2((J^2 - 1)^2) / 48\} - \{N(J^4 - 1) / 120\}. \end{aligned} \tag{31}$$

The specific example of ( $\mu_4$ ) with  $(N, J) = (2, 3)$  will be left to the reader to confirm that the formulas in equation (25) satisfy  $\sum\{(h - 2)^4 C(2, 3; h)\} = (\mu_4)(J^N) = (4)(3^2) = 36$ .

Next, the exponential generating function ( $v_{\text{egf}}$ ) for the moment about the origin is shown to have the same formula as ( $\mu_{\text{egf}}$ ) in (26), except that the summation index ( $r$ ) begins here at ( $r = 1$ ) instead of at ( $r = 2$ ).

$$v_{\text{egf}} = \text{Exp}\{N \sum\{(-1)^r (S_r)(t^r) / (r!)\}\}, \tag{32}$$

where  $S_r = (((J^r) - 1)(B_r) / r)$ , with  $B_r =$  the  $r^{\text{th}}$  Bernoulli number.

By expanding the exponential generating function ( $v_{\text{egf}}$ ), the moment equation for ( $v_R$ ) can be obtained as the coefficient of the  $((t^R) / R!)$  term. Again formula (32) was confirmed empirically to  $R = 32$ . Select cases for  $R = (1, 2, \text{ and } 4)$  will show the moment equations in terms of the distribution variables ( $N$ ) and ( $J$ ). A numerical example of ( $v_4$ ) at  $(N, J) = (2, 3)$  can illustrate the equality between calculations of the left-hand and right-hand sides of equation (24).

The first moment ( $v_1$ ) gives the value of the mean ( $M$ ) of the  $C(N, J; h)$  distribution. The mean ( $M$ ) is the midpoint of ( $h$ ) over the range  $0 \leq h \leq N(J - 1)$ , which must equal  $N(J - 1) / 2$ . And this value is identical to the moment derivation from (32) for the first moment ( $v_1$ ):

$$\begin{aligned} v_1 &= -N(S_1) = -N(J^1 - 1)B_1 / 1 \quad \text{where } B_1 = (-1/2), \\ v_1 &= N(J - 1) / 2 = M. \end{aligned} \tag{33}$$

The result for ( $v_1$ ) generalizes the known binomial formula in [10] for the summation of  $\sum\{(h)^1 C(N, J; h)\} = N(2)^{(N-1)}$ , since at  $(J = 2)$  this also equals  $(v_1)(J^N) = (N/2)(2^N) = N(2)^{(N-1)}$ .

Also, from the ( $v_{\text{egf}}$ ) expansion at  $(R = 2)$ , the second moment formula for ( $v_2$ ) becomes

$$v_2 = N^2((J-1)^2 / 4) + N((J^2 - 1) / (12)). \quad (34)$$

The binomial case for  $\sum\{(h)^2 C(N, J; h)\}$  in (24) is given at  $(J=2)$  by taking  $(v_2)$  times  $(2^N)$ , where  $(v_2) = (1/4)N^2 + (1/4)N$ . So  $\sum\{(h)^2 C(N, 2; h)\} = (v_2)(2^N) = N(N+1)(2^{N-2})$ .

The fourth moment  $(v_4)$  is obtained from the coefficient of  $((t^4)/4!)$  in the expansion of  $(v_{\text{egf}})$  in formula (32), with the Bernoulli numbers for  $(r \geq 1)$  of  $\{(-1/2), (1/6), 0, (-1/30), \dots\}$ :

$$v_4 = N^4((J-1)^4 / 16) + N^3((J-1)^2(J^2 - 1) / 8) + N^2((J^2 - 1)^2 / 48) - N((J^4 - 1) / 120). \quad (35)$$

Calculation of  $(v_4)$  for a  $C(N, J; h)$  distribution with  $(N, J) = (2, 3)$  will be left to the reader to confirm that the formulas in equation (24) satisfy  $\sum\{(h)^4 C(2, 3; h)\} = (v_4)(J^N) = (52)(3^2) = 468$ .

The mean  $(v_1)$  was shown in (33) to be equal to  $(N(J-1)/2)$ , and the variance  $(\mu_2)$  was given in (28) as  $(N(J^2 - 1)/12)$ . Now, the two exponential generating functions from (26) and (32) can be redefined in terms of the mean and variance. To make the adjustment, the summation factor  $(N)$  is multiplied by  $((J^2 - 1)/12)$  to get the variance  $(\mu_2)$ . To balance this multiplication, the terms  $(S_r)$  are then divided by the same  $((J^2 - 1)/12)$  factor, thus creating a new summation term  $(T_r)$ .

Applying this transformation to formulas (26) and (32), alternative definitions of the exponential generating functions for  $(\mu_{\text{egf}})$  and  $(v_{\text{egf}})$  become the exponential power series in (36) and (37), with sums of integer index  $(r)$  over  $2 \leq r < \infty$  for  $(\mu_{\text{egf}})$  and over  $1 \leq r < \infty$  for  $(v_{\text{egf}})$ :

$$\mu_{\text{egf}} = \text{Exp}\{(\mu_2) \sum((-1)^r (T_r)(t^r) / (r!))\}, \quad (36)$$

$$v_{\text{egf}} = \text{Exp}\{(\mu_2) \sum((-1)^r (T_r)(t^r) / (r!))\}, \quad (37)$$

where  $T_r = (12)S_r / (J^2 - 1) = \{(12)B_r / r\} \{((J^r) - 1) / (J^2 - 1)\}$  with  $B_r$  = the  $r^{\text{th}}$  Bernoulli number.

The special characteristics of the  $(T_r)$  sequence depend on the Bernoulli numbers  $(B_r)$ , by definition. Like the  $(S_r)$  sequence, the values of  $(T_{2i+1})$  are zero since  $(B_{2i+1})$  are zero for integers  $(i \geq 1)$ . Additionally, the value of  $(T_2)$  is always equal to one for all  $(J)$ .

$$T_2 = \{(12)B_2 / 2\} \{((J^2) - 1) / (J^2 - 1)\} = \{(12)B_2 / 2\} \{1\} = 1, \text{ since } B_2 = 1/6.$$

With equations (36) and (37), various moment equations such as  $(\mu_4)$  and  $(v_4)$  can be derived:

$$\mu_4 = 3(\mu_2)^2(T_2)^2 + (\mu_2)(T_4), \quad (38)$$

$$v_4 = (\mu_2)^4(-T_1)^4 + 6(\mu_2)^3(-T_1)^2(T_2) + 3(\mu_2)^2(T_2)^2 + (\mu_2)(T_4). \quad (39)$$

Since  $(\mu_2) = (N(J^2 - 1)/12)$  from equation (28), specific distributions can be evaluated exactly by knowing the values of  $(N)$  and  $(J)$ . In the fourth moment example for a  $C(N, J; h)$  distribution with  $(N, J) = (2, 3)$ , the  $(T_r)$  sequence for  $r \geq 1$  begins with  $\{(-3/2), 1, 0, (-1), \dots\}$  and  $(\mu_2) = 4/3$ . The reader is invited to confirm that the fourth moments are again  $(\mu_4 = 4)$  and  $(v_4 = 52)$ . Therefore, either interpretation of exponential generating functions, with  $S_r$  or  $T_r$ , gives the correct moment value.

The general rule for the leading term in the moment equation  $(\mu_R)$  becomes apparent by observation of equation (36). Because the odd-indexed Bernoulli numbers  $(B_r)$  are zero for  $r > 1$ , and because  $(T_2) = 1$ , the first term in the  $(\mu_R)$  moment equation will be a double factorial  $(R-1)!!$  times  $(\mu_2)$  to the power  $(R/2)$ . In the sample equation (38) for  $(\mu_4)$ , this first term was  $3(\mu_2)^2$ .

Since the summation in the exponential generating function for  $(v_{\text{egf}})$  of (37) begins at index  $(r = 1)$ , the leading term in the moment equation  $(v_R)$  is always  $(-T_1)^R$  times  $(\mu_2)^R$ . In example (39) for  $(v_4)$ , this first term was  $(-T_1)^4(\mu_2)^4$ , or simply  $(-T_1\mu_2)^4$ . But, by equation (40), the product term  $(-T_1\mu_2)$  is just equal to  $\{N(J-1)/2\}$ , which is the value of the mean  $(v_1)$ , as seen in equation (33). Therefore, the leading term in the moment equation  $(v_R)$  is always  $(v_1)^R$ .

$$(-T_1\mu_2) = \{6/(J+1)\}\{\mu_2\} = \{6/(J+1)\}\{N(J^2-1)/12\} = \{N(J-1)/2\} = (v_1), \quad (40)$$

since  $T_1 = \{(12)B_1/1\}\{((J^1)-1)/(J^2-2)\} = \{-6/(J+1)\}$  with  $B_1 = (-1/2)$ .

Also, the term  $(-T_1\mu_2)$  is the coefficient of  $((t^r)/(r!))$  at  $r = 1$  in the exponential generating function  $(v_{\text{egf}})$  from (37). This first term of the summation can now be rewritten by the equality  $(-T_1\mu_2)(t^1)/(1!) = (v_1t)$ . If this term is extracted from the summation in (37), the remaining non-zero terms in the summation have even indices in  $(r)$ , since all of the odd indexed terms have a factor that is zero; i.e., the odd indexed Bernoulli numbers  $B_{2i+1}$  for  $i \geq 1$ . Using this information, both exponential generating functions  $(\mu_{\text{egf}})$  and  $(v_{\text{egf}})$  from (36) and (37) can be rewritten with the zero-valued summation terms excluded. The exponential generating functions are now both summed for the redefined index  $(r)$  over all integers in the range  $1 \geq r > \infty$ :

$$\mu_{\text{egf}} = \text{Exp}\{(\mu_2) \sum((T_{2r})(t^{2r})/((2r)!))\}, \quad (41)$$

$$v_{\text{egf}} = \text{Exp}\{(v_1t) + (\mu_2) \sum((T_{2r})(t^{2r})/((2r)!))\}, \quad (42)$$

where  $T_{2r} = (12)S_{2r}/(J^2-1) = \{(12)B_{2r}/(2r)\}\{((J^{2r})-1)/(J^2-1)\}$  with  $B_{2r}$  = the  $(2r)^{\text{th}}$  Bernoulli number.

A comparison between moment generating functions of the discrete  $C(N, J; h)$  distribution and a normal distribution in the continuous case is useful in this format. The exponential generating functions for the continuous and normal distribution are given in [3] as:

$$\mu_{\text{egf}} \text{ (continuous \& normal)} = \text{Exp}\{\mu_2(t^2)/2\} \text{ and}$$

$$v_{\text{egf}} \text{ (continuous \& normal)} = \text{Exp}\{(v_1t) + \mu_2(t^2)/2\}.$$

Since  $(T_2)$  was shown to be equal to one, the continuous & normal (C&N) distribution and the discrete  $C(N, J; h)$  distribution have moment generating functions that have identical initial summation terms. The relationships between the moment generating functions of both of these distributions are summarized in equations (43) and (44), with the second factor  $\text{Exp}\{\Sigma\}$  being an exponential summation for integers  $2 \leq r < \infty$ :

$$\mu_{\text{egf}(C(N, J; h))} = \mu_{\text{egf}(C\&N)} \text{Exp}\{\sum((T_{2r})(t^{2r})/((2r)!))\} \quad (43)$$

and

$$v_{\text{egf}(C(N, J; h))} = v_{\text{egf}(C\&N)} \text{Exp}\{\sum((T_{2r})(t^{2r})/((2r)!))\}, \quad (44)$$

where  $T_{2r} = \{(12)B_{2r}/(2r)\}\{((J^{2r})-1)/(J^2-1)\}$ .

Besides the methodology of creating moment equations from exponential generating functions, another technique was originally used that involved recursion equations. The main recursion equation developed related the  $R^{\text{th}}$  moment  $(\mu_R)$  about the mean  $(v_1)$  to the previously generated moments  $v_{(R-h)}$  about the origin. The summation was taken over all integers of  $(h)$  in the range  $0 \leq h \leq R$ :

$$\mu_R = \sum (-1)^h \binom{R}{h} (v_1)^h (v_{R-h}). \quad (45)$$

The recursion equation (45) originated from observation of moment equations ( $\mu_R$ ) from [3] for  $R = 0$  through 4, including the oft quoted  $(\mu_2) = (v_2) - (v_1)^2$ . This latter formula is apparent from equation (34) in the form  $(v_2) = (v_1)^2 + (\mu_2)$ .

When  $R$  is an odd integer, the moment ( $\mu_R$ ) is zero. Therefore, the recursion equation can also relate successive moments ( $v_R$ ) to previous moments ( $v_{(R-h)}$ ). And now the summation is taken over all integers of ( $h$ ) in the range  $1 \leq h \leq R$ . Thus, after defining  $(v_0) = 1$ , the recursion equation for ( $v_R$ ) at ( $R = \text{odd}$ ) becomes

$$v_R = \sum (-1)^{(h+1)} \binom{R}{h} (v_1)^h (v_{(R-h)}). \quad (46)$$

## 6. SUMMATIONS OF ALTERNATING SIGNED TERMS

The alternating signed version of the moment equations yields the following summation formulas for the  $C(N, J, h)$  terms, when summed over integers  $0 \leq h \leq N(J-1)$ :

$$\sum \{(-1)^h (h)^R C(N, J, h)\} = v_{Ra}, \quad (47)$$

$$\sum \{(-1)^h (h-M)^R C(N, J, h)\} = \mu_{Ra}, \quad (48)$$

where  $M = (v_1) = N(J-1)/2$ .

Two separate presentations on the moments for the alternating signed  $C(N, J, h)$  terms will be given for cases when ( $J$ ) is an odd integer and when ( $J$ ) is an even integer. Thus, two different notations will be assigned for each case:

1.  $(v_{Ra}) = (v_{R,d})$  and  $(\mu_{Ra}) = (\mu_{R,d})$ , when  $J = \text{odd}$ ;
2.  $(v_{Ra}) = (v_{R,e})$  and  $(\mu_{Ra}) = (\mu_{R,e})$ , when  $J = \text{even}$ .

Compared to equations (24) and (25) of the previous moment section, the most apparent difference in the summation equations of (47) and (48) is that they do not have the common factor ( $J^N$ ).

In the initial definition of the moment equations (5), the common factor to be multiplied by ( $v_{Ra}$ ) and ( $\mu_{Ra}$ ) was the sum of all the distribution terms in the  $N^{\text{th}}$  row of the Pascal-de Moivre triangle. From the discussion on quadruplet cycles, the difference of the two alternate term equations ( $A$  minus  $B$ ) from equations (19) and (20) will give the sum of the alternating signed  $C(N, J, h)$  distribution. For all cases when ( $J$ ) is odd, it has a value of one:

$$\sum \{(-1)^h C(N, J, h)\} = 1 \text{ for } J = \text{odd}. \quad (49)$$

When ( $J$ ) is an odd integer, the moment equations for ( $v_{R,d}$ ) and ( $\mu_{R,d}$ ) are derived directly from their exponential generating functions: ( $\mu_{\text{egf},d}$ ) and ( $v_{\text{egf},d}$ ). The exponential generating functions are expanded to obtain ( $\mu_{R,d}$ ) and ( $v_{R,d}$ ) as coefficients of the term  $((t^R)/R!)$  in the summation over integers  $0 \leq i < \infty$ , as outlined in [3]:

$$\begin{aligned} \mu_{\text{egf},d} &= \sum (\mu_{i,d})(t^i) / (i!); \\ v_{\text{egf},d} &= \sum (v_{i,d})(t^i) / (i!). \end{aligned}$$

In this ( $J = \text{odd}$ ) case, the exponential generating functions for  $(\mu_{\text{egf},d})$  and  $(v_{\text{egf},d})$  turn out to have identical formulas, as expressed by the exponential power series in (50) and (51). Just the ranges of their summation index ( $r$ ) differ, with  $2 \leq r < \infty$  for  $(\mu_{\text{egf},d})$  and  $1 \leq r < \infty$  for  $(v_{\text{egf},d})$ :

$$\mu_{\text{egf},d} = \text{Exp}\{N \sum((-1)^r (S_{r,d})(t^r) / (r!))\}, \tag{50}$$

$$v_{\text{egf},d} = \text{Exp}\{N \sum((-1)^r (S_{r,d})(t^r) / (r!))\}, \tag{51}$$

where  $S_{r,d} = ((2^r) - 1)S_r = ((2^r) - 1)((J^r) - 1)(B_r) / r$  with  $B_r =$  the  $r^{\text{th}}$  Bernoulli number.

These exponential generating functions  $(\mu_{\text{egf},d})$  and  $(v_{\text{egf},d})$  for these alternating signed distributions are closely related to the exponential generating functions  $(\mu_{\text{egf}})$  and  $(v_{\text{egf}})$  of (26) and (32) for the positively signed distributions. Only the summation term  $(S_{r,d})$  in (50) and (51) differs from the summation term  $(S_r)$  in (26) and (32) by a factor of  $((2^r) - 1)$ .

The same transformation methods described in (36), (37), (41), and (42) will change the summation variables ( $N$ ) and  $(S_{r,d})$  in equations (50) and (51) into the variables  $(\mu_2)$  and  $(T_{r,d})$  for equations (52) and (53). These new exponential generating functions are both summed over all integers in the range  $1 \leq r < \infty$ . Note:  $(v_{1,d}) = v_1$  and  $(\mu_{2,d}) = 3\mu_2$ .

$$\mu_{\text{egf},d} = \text{Exp}\{(\mu_2) \sum((T_{2r,d})(t^{2r}) / ((2r)!))\}, \tag{52}$$

$$v_{\text{egf},d} = \text{Exp}\{(v_1 t + (\mu_2) \sum((T_{2r,d})(t^{2r}) / ((2r)!))\}, \tag{53}$$

where  $T_{2r,d} = (12)S_{2r,d} / (J^2 - 1) = \{(2^{2r}) - 1\} \{(12)B_{2r} / (2r)\} \{(J^{2r}) - 1\} / (J^2 - 1)$  with  $B_{2r} =$  the  $(2r)^{\text{th}}$  Bernoulli number.

The fourth moments  $(\mu_{4,d})$  and  $(v_{4,d})$ , for example, can be obtained from the coefficient of  $((t^4) / 4!)$  in the expansions of  $(\mu_{\text{egf},d})$  and  $(v_{\text{egf},d})$  in formulas (52) and (53). The format of these formulas is the same as for  $(\mu_4)$  and  $(v_4)$  in (38) and (39), where the definition of each summation term  $(T_{r,d})$  has only changed from  $(T_r)$  by a factor  $((2^r) - 1)$ .

$$\mu_{4,d} = 3(\mu_2)^2(T_{2,d})^2 + (\mu_2)(T_{4,d}), \tag{54}$$

$$v_{4,d} = (\mu_2)^4(-T_{1,d})^4 + 6(\mu_2)^3(-T_{1,d})^2(T_{2,d}) + 3(\mu_2)^2(T_{2,d})^2 + (\mu_2)(T_{4,d}). \tag{55}$$

A specific fourth moment example for  $(\mu_{4,d})$  and  $(v_{4,d})$  of a distribution with  $(N, J) = (2, 3)$  can be calculated by inserting the proper values for  $(\mu_2)$  and  $(T_{r,d})$  in (54) and (55). Here, the  $(T_{r,d})$  sequence for  $(r \geq 1)$  begins at  $\{(-3/2), 3, 0, (-15), \dots\}$ , and  $(\mu_2) = (N(J^2 - 1) / 12) = (4/3)$ . Thus, for  $(N, J) = (2, 3)$ , the reader can find that equations (54) and (55) agree with the moment values in the formulas from equations (47) and (48) to give  $(\mu_{4,d}) = 28$  and  $(v_{4,d}) = 140$ .

Now for the other type of distribution, with  $(J)$  as an even integer. In this case, the sum of the alternating signed  $C(N, J; h)$  distribution is zero for  $N \geq 1$ , as in the binomial ( $J = 2$ ) case from [10]. The left-hand side of equations (47) and (48) does exist. The interpretation of the terms  $(\mu_{R,e})$  and  $(v_{R,e})$  on the right-hand side may not be clear, since these moments and the distribution density  $f(h)$  from equation (4) have a denominator of zero; thus, they may be undefined. For convenience, these terms  $(\mu_{R,e})$  and  $(v_{R,e})$  will still be called moments, in the sense that they are being used to generate the summation formulas in (47) and (48).

The simplest results for the  $R^{\text{th}}$  moment equations  $(\mu_{R,e})$  and  $(v_{R,e})$  for the  $(J = \text{even})$  cases occur when  $(N)$  is greater than  $(R)$ . Then we do get moment equations of zero:

$$\mu_{R,e} = 0 \quad \text{and} \quad \nu_{R,e} = 0 \quad \text{for } N > R. \quad (56)$$

However, for  $(R \geq N)$ , the moment values are predominantly nonzero, when  $(J)$  is an even integer. The moment equations for  $(\nu_{R,e})$  and  $(\mu_{R,e})$  can be derived if they are broken down into two parts: a common factor (CF) and an equation  $(m_k)$  from the corresponding exponential generating functions for  $(\mu)$  and  $(\nu)$ . The moment equations are:

$$\mu_{R,e} = (\text{CF})(m_k(\mu)), \quad (57)$$

and

$$\nu_{R,e} = (\text{CF})(m_k(\nu)), \quad (58)$$

where  $(\text{CF}) = \{(-1)^N (J/2)^N (N!)\} \binom{R}{N}$ .

The equation for (CF) was obtained by empirical analysis. The equations  $(m_k)$  are the coefficients of the term  $((t^k)/k!)$  in the expansion of the exponential generating functions, where the value of  $(k)$  is defined as  $(R - N)$ . The summations in the exponential generating functions for the  $(m_k)$  coefficients are taken over all integers  $r \geq a$ , with  $a = 2$  for  $(\mu_{R,e})$  and at  $a = 1$  for  $(\nu_{R,e})$ .

$$\mu_{\text{egf},e} = \text{Exp}\{N \sum ((-1)^r (S_{r,e})(t^r)/(r!))\} \quad \text{for } 2 \leq r < \infty, \quad (59)$$

$$\nu_{\text{egf},e} = \text{Exp}\{N \sum ((-1)^r (S_{r,e})(t^r)/(r!))\} \quad \text{for } 1 \leq r < \infty, \quad (60)$$

where  $S_{r,e} = \{(J^r) - (2^r) + 1\} \{(B_r)/r\}$  with  $B_r =$  the  $r^{\text{th}}$  Bernoulli number.

Again a useful transformation takes summation variables  $(N)$  and  $(S_{r,e})$  in (59) and (60) into the variables  $(\mu_2)$  and  $(T_{r,e})$  for equations (61) and (62). Both of these new exponential generating functions have summations taken for all integers  $1 \geq r > \infty$ . Note:  $(\nu_{1,e}) = \nu_1$ .

$$\mu_{\text{egf},e} = \text{Exp}\{(\mu_2) \sum ((T_{2r,e})(t^{2r})/((2r)!))\}, \quad (61)$$

$$\nu_{\text{egf},e} = \text{Exp}\{(\nu_1 t) + (\mu_2) \sum ((T_{2r,e})(t^{2r})/((2r)!))\}, \quad (62)$$

where  $T_{2r,e} = (12)S_{2r,e}/(J^2 - 1) = \{(J^{2r}) - (2^{2r}) + 1\} \{(12)/(J^2 - 1)\} \{(B_{2r})/(2r)\}$ .

The reader is challenged to find the  $(\mu_{R,e})$  moment at  $(R=4)$  for the alternating signed  $C(N, J; h)$  distribution with  $(N, J) = (2, 2)$ . In this example, the moment equation (57) has an index of  $(k = R - N = 2)$ , with factors  $(\text{CF}) = 12$  and  $m_k(\mu) = 1/6$ , which give a value for  $(\mu_{R,e})$  equal to 2.

**A Final Note:** Conditions involving distribution symmetry and the patterns of Bernoulli numbers in the exponential generating functions favor the proposition that all of the moment equations and their generating functions will remain valid for all possible  $C(N, J; h)$  moment summations with distribution variables  $(N, J)$  and moment numbers  $(R)$  over all positive integers. Extensive empirical evidence suggests optimism.

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