A NOTE ON TWO THEOREMS OF MELHAM AND SHANNON

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1. INTRODUCTION AND PRELIMINARIES

In this note we use some properties of the Lucas sequences,

$$U_n(m, Q) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n(m, Q) = \alpha^n + \beta^n, \tag{1.1}$$

where $\alpha > \beta$, $m = \alpha + \beta$, and $Q = \alpha\beta$, to extend two theorems due to Melham and Shannon [3].

For the sequences defined above, it is known that

$$U_n[V_h(m, Q), Q^h] = U_{nh}(m, Q) / U_h(m, Q) \quad (h \neq 0) \tag{1.2}$$

and

$$V_n[V_h(m, Q), Q^h] = V_{nh}(m, Q). \tag{1.3}$$

In this note we are concerned with sequences where $Q = \pm 1$. In this case, for proofs of (1.2) and (1.3) in the literature see, for example, [1, p. 632]. In [3], Melham and Shannon proved that

$$\sum_{j=1}^{\infty} \frac{1}{U_{kj}(m, 1) U_{k(j+1)}(m, 1)} = \frac{1}{\alpha^k U_k^2(m, 1)} \quad (k \neq 0) \tag{1.4}$$

and

$$\sum_{j=0}^{\infty} \frac{1}{V_{kj}(m, 1) V_{k(j+1)}(m, 1)} = \frac{1}{2(\alpha - \beta) U_k(m, 1)}. \tag{1.5}$$

They evaluated analogous sums involving $U_n(m, -1)$ and $V_n(m, -1)$ only in the special case in which $m = 1$ (Fibonacci and Lucas numbers, see (3.9) and (3.10) of [3]). The aim of this note is to extend (1.4) and (1.5) to even-subscripted numbers $U_n(m, -1)$ and $V_n(m, -1)$, with m arbitrary, so that (3.9) and (3.10) of [3] will emerge as special cases of our results.

2. OUR RESULTS

Theorem 1:
$$\sum_{j=1}^{\infty} \frac{1}{U_{2kj}(m, -1) U_{2k(j+1)}(m, -1)} = \frac{1}{\alpha^{2k} U_{2k}^2(m, -1)} \quad (k \neq 0). \tag{2.1}$$

Theorem 2:
$$\sum_{j=0}^{\infty} \frac{1}{V_{2kj}(m, -1) V_{2k(j+1)}(m, -1)} = \frac{1}{2(\alpha - \beta) U_{2k}(m, -1)}. \tag{2.2}$$

Proof of Theorem 1: If we let $U_{kt}[V_2(m, -1), 1] = U_{kt}(\overline{m}, 1)$ with $\overline{m} = \gamma + \delta$, $\gamma\delta = 1$, $\gamma > \delta$, then (1.2) may be written as

$$U_{2kt}(m, -1) = U_2(m, -1) \cdot U_{kt}(\overline{m}, 1),$$

and it follows (for $t = 1, j$ and $j + 1$) that

$$\sum_{j=1}^{\infty} \frac{1}{U_{2kj}(m,-1)U_{2k(j+1)}(m,-1)} = \frac{1}{U_2^2(m,-1)} \sum_{j=1}^{\infty} \frac{1}{U_{kj}(\overline{m},1)U_{k(j+1)}(\overline{m},1)}$$

which, by (1.4) and (1.2),

$$= \frac{1}{U_2^2(m,-1)} \cdot \frac{1}{\gamma^k U_k^2(\overline{m},1)} = \frac{1}{\gamma^k U_{2k}^2(m,-1)}.$$

Now, since $\gamma + \delta = \overline{m} = V_2(m,-1) = \alpha^2 + \beta^2$, with $\alpha\beta = -1$, we have

$$\gamma + \frac{1}{\gamma} = \alpha^2 + \frac{1}{\alpha^2},$$

whence $\gamma = \alpha^2$. This completes the proof.

By using (1.3), the proof of Theorem 2 can be carried out in a similar way, so it is left as an exercise for the interested reader.

We shall conclude this note by working out some reciprocal sums emerging from particular choices of m in (2.1) and (2.2). If we let $m = 1$, we obtain (3.9) and (3.10) of [3], respectively. If we let $m = 2$, we obtain, respectively,

$$\sum_{j=1}^{\infty} \frac{1}{P_{2kj}P_{2k(j+1)}} = \frac{1}{P_{2k}^2(3+2\sqrt{2})^k} \tag{2.3}$$

and

$$\sum_{j=0}^{\infty} \frac{1}{Q_{2kj}Q_{2k(j+1)}} = \frac{1}{4\sqrt{2}P_{2k}}, \tag{2.4}$$

where P_k (resp. Q_k) denotes the k^{th} Pell (resp. Pell-Lucas [2]) number.

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