A SPARSE MATRIX AND THE CATALAN NUMBERS*

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(Submitted May 1996-Final Revision August 1997)

1. INTRODUCTION

We shall consider a stack of r glass plates. A light ray comes from the upper left direction, reflecting at some inner boundary surfaces of the plates and passing through others. After repeated reflections and transmissions, the light ray goes away to the upper-right or the lower-right direction. How many possible paths are there in this case? The closed formulas for coefficients in the recurrent relations arising from the problem of enumeration of the possible reflection paths of light rays in the multiple glass plates were first given by J. A. Brooks (cf. [1, p. 271, eq. $T(n)$]). Using the signed ballot numbers $D(k, j)$, which are defined below, we can also obtain the formulas ([5, p. 385, eq. (3.17)]). A matrix $B = B^{(r)}$ constructed using the numbers $D(k, j)$ in a particular but natural manner indicates some interesting properties; for instance, "the sparseness" in the sense that the number of zero-elements of the matrix is maximum among the equivalent matrices. Let B^T be the transpose of B. Then the Catalan numbers (cf. [3]) appear in the matrix product of B^T and B.

The contents of this paper are regarded as continuations of [5]. For completeness, we will now summarize the results of [5].

Let A be an r by r matrix such that

$$A = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 \\ 0 & 0 & 0 & \dots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \dots & 1 & 1 & 1 \\ 0 & 1 & 1 & \dots & 1 & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 & 1 \end{pmatrix}.$$

(This matrix arises when one enumerates the increased numbers of paths of light rays produced by an extra reflection from r plates, in an iterative scheme (cf. [5].)

Then we have

$$A^{-1} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & -1 & 1 \\ 0 & 0 & 0 & \dots & -1 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & -1 & 1 & \dots & 0 & 0 & 0 \\ -1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

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* This paper, presented at the Seventh International Research Conference held in Graz, Austria, in July of 1996, was scheduled to appear in the Conference Proceedings. However, due to refereeing problems and deadline dates, we are publishing it in this issue of *The Fibonacci Quarterly* to assure its timely publication.

Let $\underline{1}$ be a column vector of size r such that

$$\underline{1}^T = (1, 1, 1, \ldots, 1, 1, 1).$$

Then successive multiplications by A^{-1} give the following sequences:

$$\begin{aligned}
\underline{1}^T A^{-1} &= (0, \ldots, 0, D(1, 0)), \\
\underline{1}^T A^{-2} &= (D(2, 0), 0, \ldots, 0), \\
&\ldots, \\
&\ldots, \\
\underline{1}^T A^{-2m+1} &= (0, \ldots, 0, D(2m-1, 0), D(2m-1, 1), \ldots, D(2m-1, m-1)), \\
\underline{1}^T A^{-2m} &= (D(2m, m-1), D(2m, m-2), \ldots, D(2m, 0), 0, \ldots, 0), \\
&\ldots,
\end{aligned}$$

where

$$D(1, 0); D(2, 0); D(3, 0), D(3, 1); D(4, 0), D(4, 1); D(5, 0), D(5, 1), D(5, 2);$$
$$\ldots = 1; 1; -1, 1; -1, 2; 1, -3, 2; \ldots, \text{ respectively.}$$

From the process used to produce $D(k, j)$, we can obtain the following recurrence relations (cf. [5, p. 382, eqs. (2.1)-(2.3)]):

$$D(k, j) = \begin{cases} (-1)^k \{D(k-1, j) - D(k-1, j-1)\} & \text{for } 1 \le j \le \lfloor \frac{k-1}{2} \rfloor, \\ (-1)^{\lfloor \frac{k-1}{2} \rfloor} & \text{for } j = 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\lfloor x \rfloor$ is the floor function of D. Knuth and represents the greatest integer less than or equal to x (see [4]). Hence, we can get a closed expression for the numbers $D(k, j)$ $(1 \le k; 0 \le j \le \lfloor (k-1)/2 \rfloor)$, namely,

$$D(k, j) = (-1)^{\lfloor \frac{k-1}{2} \rfloor + j} \frac{k-2j}{k} \binom{k}{j} \tag{1}$$

(cf. [5, p. 382, eq. (2.6)]). The ballot numbers can be expressed as

$$\mathrm{bal}(k, j) = \frac{k-2j}{k} \binom{k}{j}$$

(cf. [2, p. 73]). So our numbers are called "signed ballot numbers." The Catalan numbers c_n are usually defined as

$$c_n = \frac{1}{n+1} \binom{2n}{n}.$$

In particular, for both even and odd cases, if $k = 2k'$ and $k = 2k'+1$, respectively, we have

$$D(2k', k'-1) = D(2k'+1, k') = \frac{1}{k'+1} \binom{2k'}{k'} = c_{k'}.$$

Hence, we can regard our numbers $\{D(k, j)\}$ as signed ballot numbers and, simultaneously, as a generalization of the Catalan numbers.

Let B be a matrix such that

$$B = (A^{-1}\underline{1}, A^{-2}\underline{1}, \dots, A^{-(r-1)}\underline{1}, A^{-r}\underline{1}). \tag{2}$$

(In [5], we use the symbol B^T in place of B (see [5, p. 381]).

It can be shown that the Catalan numbers c_n and zeros appear alternately in the first row and the last row of B (cf. [5, p. 382, eq. (2.7)], and see B below for the case $r = 9$).

For $m = \dots, -2, -1, 0, 1, 2, \dots$, let us consider an associated set of linear equations, that is, $(A^{m-1}\underline{1}, A^{m-2}\underline{1}, \dots, A^{m-r}\underline{1})\underline{x} = A^m\underline{1}$. (This $\underline{x}$ is the coefficient vector of the recurrent relations arising from the problem of light rays in multiple glass plates (cf. [5]).) Then the matrix B is the coefficients matrix for the case $m = 0$, from which we can obtain the solution $\underline{x} = B^{-1}\underline{1}$, where B is a nonsingular matrix because of (7) below.

Let $T_n = T_n^{(r)}$ be the total number of ray paths formed by the r plates after n reflections, and let $\underline{t} = \underline{t}^{(r)} = (T_{n-1}, T_{n-2}, \dots, T_{n-r})^T$. It is shown in [1, p. 271] and [5, p. 385, eq. (3.17)] that

$$T_n = (B^{-1}\underline{1})^T\underline{t} = \sum_{j=1}^{r}(-1)^{\lfloor\frac{j-1}{2}\rfloor}\binom{\lfloor\frac{r-j}{2}\rfloor + j}{j}T_{n-j}.$$

For the (p, q) element $z_{p,q}$ of B^{-1}, we notice that the following are also valid:

$$\begin{cases} z_{2p',\,p'-1+m} = (-1)^{p'-1}\dbinom{2p'-2+m}{2p'-1} \dots 1 \le p' \le \lfloor r/2\rfloor, 1 \le m \le \lfloor r/2\rfloor - p' + 1, \\ z_{2p'+1,\lfloor r/2\rfloor+m} = (-1)^{p'}\dbinom{\lfloor (r+1)/2\rfloor + p' - m}{2p'} \dots 0 \le p' \le \lfloor (r-1)/2\rfloor, 1 \le m \le \lfloor (r+1)/2\rfloor - p', \\ z_{p,q} = 0 \dots \text{otherwise}. \end{cases} \tag{3}$$

(See [5, eqs. (3.8)-(3.10)].) An algebraic manipulation yields

$$z_{p,q} = \begin{cases} (-1)^{p/2-1}\dbinom{p/2+q-1}{p-1} & \text{for } p \text{ even; } \ p/2 \le q \le \lfloor r/2\rfloor, \\ (-1)^{\lfloor p/2\rfloor}\dbinom{r+\lfloor p/2\rfloor - q}{p-1} & \text{for } p \text{ odd; } \ \lfloor r/2\rfloor + 1 \le q \le r - \lfloor p/2\rfloor, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

For example, in the case $r = 9$, we have

$$B = \begin{pmatrix} 0 & 1 & 0 & 2 & 0 & 5 & 0 & 14 & 0 \\ 0 & 0 & 0 & -1 & 0 & -4 & 0 & -14 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 0 & 5 & 0 & 20 \\ 0 & 0 & -1 & 0 & -3 & 0 & -9 & 0 & -28 \\ 1 & 0 & 1 & 0 & 2 & 0 & 5 & 0 & 14 \end{pmatrix}.$$

and

$$B^{-1}=\begin{pmatrix}0&0&0&0&1&1&1&1&1\\1&2&3&4&0&0&0&0&0\\0&0&0&0&-10&-6&-3&-1&0\\0&-1&-4&-10&0&0&0&0&0\\0&0&0&0&15&5&1&0&0\\0&0&1&6&0&0&0&0&0\\0&0&0&0&-7&-1&0&0&0\\0&0&0&-1&0&0&0&0&0\\0&0&0&0&1&0&0&0&0\end{pmatrix}.$$

2. CATALAN NUMBERS IN B^TB

Now we will discuss further properties of B. For matrix B, computing B^TB, we have the Catalan numbers and zeros that run parallel to the skew-diagonal line. From the lower-left to the upper-right of B^TB, the numbers $c_0, c_1, \dots, c_n$ appear on the first, the third, ..., and the $(2n+1)^{\text{st}}$ line, respectively; i.e., we have

$$B^TB=\begin{pmatrix}c_0&0&c_1&0&c_2&\cdots&0&c_{r'}\\0&c_1&0&c_2&0&\cdots&c_{r'}&0\\c_1&0&c_2&0&c_3&\cdots&0&c_{r'+1}\\0&c_2&0&c_3&0&\cdots&c_{r'+1}&0\\c_2&0&c_3&0&c_4&\cdots&0&c_{r'+2}\\0&c_3&0&c_4&0&\cdots&c_{r'+2}&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&c_{r'}&0&c_{r'+1}&0&\cdots&c_{2r'-1}&0\\c_{r'}&0&c_{r'+1}&0&c_{r'+2}&\cdots&0&c_{2r'}\end{pmatrix},$$

where $r=2r'+1$. In the case $r=2r'$, to obtain the expression B^TB, we have to delete the last row and the last column from the one above. All the odd skew-diagonal elements of order $2n+1$ running from the lower-left to the upper-right of B^TB are the Catalan number c_n, while those of even order are zero. Namely, we have the following theorem.

Theorem 1: For every k $(1\le k\le r)$, it holds that

$$(B^TB)_{i,j}=\begin{cases}c_{k-1} & \text{for } (i,j)=(k+m,k-m) \text{ and } (k-m,k+m),\\ 0 & \text{otherwise,}\end{cases}$$

where

$$m=\begin{cases}0,1,\dots,k-2 & \text{for } 2\le k\le\lfloor\frac{r+1}{2}\rfloor,\\ 0,1,\dots,r-k & \text{for } \lfloor\frac{r+1}{2}\rfloor+1\le k\le r.\end{cases}$$

Proof: From (2), consider an odd-skew-diagonal matrix element, we deal with the two cases simultaneously:

$$\begin{aligned}(B^TB)_{k\pm m,\,k\mp m}&=\underline{1}^TA^{-k\mp m}A^{-k\pm m}\underline{1}=\underline{1}^TA^{-2k}\underline{1}=\underline{1}^TA^{-2k+1}A^{-1}\underline{1}\\&=(0,\dots,0,D(2k-1,0),\dots,D(2k-1,k-2),D(2k-1,k-1))(0,\dots,0,c_0)^T\\&=D(2k-1,k-1)c_0=c_{k-1}.\end{aligned}$$

Next, consider an even-skew-diagonal matrix element:

$$(B^TB)_{k\pm(m+1),\, k\mp m} = \underline{1}^T A^{-k\mp(m+1)} A^{-k\pm m}\underline{1} = \underline{1}^T A^{-2k\mp 1}\underline{1}.$$

In the upper sign case, we have

$$\begin{aligned} &= \underline{1}^T A^{-2k} A^{-1}\underline{1} \\ &= (D(2k, k-1), D(2k, k-2), \ldots, D(2k, 0), 0, \ldots, 0)(0, \ldots, 0, c_0)^T \\ &= 0, \end{aligned}$$

where

$$m = \begin{cases} 0, 1, \ldots, (\cdot) & \text{for } 1 \le k \le \lfloor \frac{r+1}{2} \rfloor, \\ 0, 1, \ldots, r-k-1 & \text{for } \lfloor \frac{r+1}{2} \rfloor + 1 \le k \le r-1, \end{cases}$$

where

$$(\cdot) = \begin{cases} k-2 & \text{for } r = 2r'+1, \\ k-1 & \text{for } r = 2r'. \end{cases}$$

In the lower sign case, we have

$$\begin{aligned} &= \underline{1}^T A^{-2k+2} A^{-1}\underline{1} \\ &= (D(2k-2, k-2), D(2k-2, k-3), \ldots, D(2k-2, 0), 0, \ldots, 0)(0, \ldots, 0, c_0)^T \\ &= 0, \end{aligned}$$

where

$$m = \begin{cases} 0, 1, \ldots, k-2 & \text{for } 2 \le k \le \lfloor \frac{r+1}{2} \rfloor, \\ 0, 1, \ldots, r-k & \text{for } \lfloor \frac{r+1}{2} \rfloor + 1 \le k \le r. \end{cases}$$

This establishes Theorem 1.

As a corollary, we also have, from (1):

$$\begin{aligned} (B^TB)_{k\pm m,\, k\mp m} &= \sum_{j=0}^{\lfloor (k-m-1)/2 \rfloor} D(k\pm m, m+j) D(k\mp m, j) \\ &= \frac{1}{k^2-m^2} \sum_{j=0}^{\lfloor (k-m-1)/2 \rfloor} (k\mp m-2j)^2 \binom{k\pm m}{k-j}\binom{k\mp m}{j} \\ &\equiv c_{k-1}, \end{aligned}$$

where

$$m = \begin{cases} 0, 1, \ldots, k-1 & \text{for } 1 \le k \le \lfloor \frac{r+1}{2} \rfloor, \\ 0, 1, \ldots, r-k & \text{for } \lfloor \frac{r+1}{2} \rfloor + 1 \le k \le r. \end{cases}$$

This is a binomial identity for the Catalan numbers.

For example, in the case $r = 9$, we have

$$B^TB = \begin{pmatrix} 1 & 0 & 1 & 0 & 2 & 0 & 5 & 0 & 14 \\ 0 & 1 & 0 & 2 & 0 & 5 & 0 & 14 & 0 \\ 1 & 0 & 2 & 0 & 5 & 0 & 14 & 0 & 42 \\ 0 & 2 & 0 & 5 & 0 & 14 & 0 & 42 & 0 \\ 2 & 0 & 5 & 0 & 14 & 0 & 42 & 0 & 132 \\ 0 & 5 & 0 & 14 & 0 & 42 & 0 & 132 & 0 \\ 5 & 0 & 14 & 0 & 42 & 0 & 132 & 0 & 429 \\ 0 & 14 & 0 & 42 & 0 & 132 & 0 & 429 & 0 \\ 14 & 0 & 42 & 0 & 132 & 0 & 429 & 0 & 1430 \end{pmatrix}.$$

We may remark here that BB^T is a particular kind of block matrix, with symmetric blocks in the main diagonal. For example, in the case $r = 9$, we have

$$BB^T = \begin{pmatrix} 226 & -218 & 89 & -14 & 0 & 0 & 0 & 0 & 0 \\ -218 & 213 & -88 & 14 & 0 & 0 & 0 & 0 & 0 \\ 89 & -88 & 37 & -6 & 0 & 0 & 0 & 0 & 0 \\ -14 & 14 & -6 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -7 & 20 & -28 & 14 \\ 0 & 0 & 0 & 0 & -7 & 50 & -145 & 205 & -103 \\ 0 & 0 & 0 & 0 & 20 & -145 & 426 & -608 & 307 \\ 0 & 0 & 0 & 0 & -28 & 205 & -608 & 875 & -444 \\ 0 & 0 & 0 & 0 & 14 & -103 & 307 & -444 & 227 \end{pmatrix}.$$

3. SPARSENESS OF B AND B^{-1}

We may call B "a sparse matrix" (for A) in the sense that, for a regular matrix A, it holds that

$$\max_{m:\,\text{integer}} n\{A^m B\} = n\{B\}$$

and, simultaneously, that

$$\max_{m:\,\text{integer}} n\{B^{-1}A^m\} = n\{B^{-1}\}$$

where $n\{M\}$ is the number of zero-elements of a matrix (or vector) M. We shall establish below that

$$n\{B\} = n\{B^{-1}\} = \lfloor \tfrac{r}{2} \rfloor \left(3\lfloor \tfrac{r+1}{2} \rfloor - 1\right),$$
$$|B| = |B^{-1}| = (-1)^{\lfloor r/2 \rfloor}.$$

To prove these statements, we need the following lemma.

Lemma: For nonnegative integers $m \ge 0$, we have

$$n\{A^{-m}B\} = \begin{cases} \lfloor \frac{r}{2} \rfloor (3\lfloor \frac{r+1}{2} \rfloor - 1) - r\lfloor \frac{m}{2} \rfloor & \text{for } m = 2m', \\ \lfloor \frac{r}{2} \rfloor (3\lfloor \frac{r+1}{2} \rfloor - 2) - r\lfloor \frac{m}{2} \rfloor & \text{for } m = 2m'+1. \end{cases} \tag{5}$$

Proof of the Lemma: From the expression $\underline{1}^T A^{-m}$ in Section 1, it follows by inspection that

$$n\{A^n \underline{1}\} = 0 \qquad \text{for all } n \ge 0,$$
$$n\{A^{-1}\underline{1}\} = r - 1,$$
$$n\{A^{-2}\underline{1}\} = r - 1,$$

$$n\{A^{-m}\underline{1}\} = r - \left\lfloor \tfrac{m+1}{2} \right\rfloor,$$
$$n\{A^{-m^*}\underline{1}\} = r - \left\lfloor \tfrac{m^*+1}{2} \right\rfloor = 0,$$

where $m^* = 2r-1$. Hence, for $m \geq 0$, we have

$$\begin{aligned} n\{A^{-m}B\} &= n\{(A^{-m-1}\underline{1}, A^{-m-2}\underline{1}, \dots, A^{-m-r}\underline{1})\} \\ &= \sum_{k=1}^{r} n\{A^{-m-k}\underline{1}\} = \sum_{k=1}^{r} \left(r - \left\lfloor \tfrac{m+k+1}{2} \right\rfloor\right). \end{aligned}$$

To establish the Lemma, we may calculate the last summation separately for the even and odd cases of both m and r.

First, in the case in which $m = 2m'$, we obtain the following results:

(i) When $r = 2r'+1$, we get

$$\begin{aligned} n\{A^{-m}B\} &= r^2 + m' + r' + 1 - 2(m'+1+\cdots+m'+r'+m'+r'+1) \\ &= 3r'^2 + 2r' - m'(2r'+1) = \left\lfloor \tfrac{r}{2} \right\rfloor \left(3\left\lfloor \tfrac{r+1}{2} \right\rfloor - 1\right) - r\left\lfloor \tfrac{m}{2} \right\rfloor. \end{aligned}$$

(ii) When $r = 2r'$, we get

$$\begin{aligned} n\{A^{-m}B\} &= r^2 - 2(m'+1+\cdots+m'+r') \\ &= 3r'^2 - r' - 2r'm' = \left\lfloor \tfrac{r}{2} \right\rfloor \left(3\left\lfloor \tfrac{r+1}{2} \right\rfloor - 1\right) - r\left\lfloor \tfrac{m}{2} \right\rfloor. \end{aligned}$$

The case in which $m = 2m'+1$ is derived in an analogous fashion, so we omit the discussion for brevity. This proves the Lemma.

We now have the following theorem.

Theorem 2:

(a) For the r by r matrix B, we have

$$n\{B\} = \left\lfloor \tfrac{r}{2} \right\rfloor \left(3\left\lfloor \tfrac{r+1}{2} \right\rfloor - 1\right), \tag{6}$$

$$|B| = (-1)^{\lfloor \frac{r}{2} \rfloor}, \tag{7}$$

$$\max_{m:\,\text{integer}} n\{A^m B\} = n\{B\}. \tag{8}$$

(b) For the matrix B^{-1}, we have

$$n\{B^{-1}\} = \left\lfloor \tfrac{r}{2} \right\rfloor \left(3\left\lfloor \tfrac{r+1}{2} \right\rfloor - 1\right), \tag{9}$$

$$|B^{-1}| = (-1)^{\lfloor \frac{r}{2} \rfloor}, \tag{10}$$

$$\max_{m:\,\text{integer}} n\{B^{-1}A^m\} = n\{B^{-1}\}. \tag{11}$$

Proof of (a): In (5) of the Lemma, putting $m = 0$, we immediately have (6).

For (7), the proof is by induction. If $r = 2$, $B^{(2)}$ is a skew unit matrix of order 2. Hence, we have $|B^{(2)}| = -1$. Here, we note that in order to construct $B^{(r+1)}$ of order $r+1$ from $B^{(r)}$ of order

r, we must affix the column vector $A^{-r-1}\underline{1}$ of size $r+1$ to $B^{(r)}$ as the last column, and also affix the row vector $(0, 0, \ldots, 0, D(r+1, 0))$ of size $r+1$ to $B^{(r)}$ as the central row. Using Laplace's expansion theorem, we have

$$|B^{(r+1)}| = (-1)^{\lfloor (r+2)/2 \rfloor + r+1} D(r+1, 0) |B^{(r)}| = (-1)^{\lfloor (r+1)/2 \rfloor}.$$

Thus, we have the desired result.

For (8), from (2) for l $(1 \le l \le r)$, we have

$$n\{A^l B\} = n\{(A^{l-1}\underline{1}, A^{l-2}\underline{1}, \ldots, \underline{1}, \ldots, A^{l-r}\underline{1})\}.$$

Since columns of $A^l B$ for which the exponent of A^{l-s} is nonnegative $(1 \le s \le l)$ have no zero-elements, we have

$$\begin{aligned} n\{B\} > n\{AB\} > n\{A^2 B\} > \cdots > n\{A^r B\} \\ = n\{A^{r+1} B\} = n\{A^{r+2} B\} = \cdots = 0. \end{aligned}$$

On the other hand, by virtue of the Lemma, we have

$$\begin{aligned} n\{B\} > n\{A^{-1} B\} > n\{A^{-2} B\} > \cdots > n\{A^{-r^*+1} B\} \\ = n\{A^{-r^*} B\} = n\{A^{-r^*-1} B\} = \cdots = 0, \end{aligned}$$

where

$$r^* = \begin{cases} 4r' + 1 \ldots & \text{for } r = 2r' + 1, \\ 4r' - 1 \ldots & \text{for } r = 2r'. \end{cases}$$

This proves (a).

Proof of (b): For (9), from the available range of each subscript in the expression for the elements of B^{-1} [see (4) above], we can count the number $n\{B^{-1}\}$ of zero-elements of B^{-1}.

The validity of (10) follows from (7).

To establish (11), we must count the number of zero-elements of $B^{-1}A^m$. Let L, C, and R be the number of zero-elements of $B^{-1}A^m$ $(0 \le m \le r-1)$ in the left parts $(1 \le j \le \lfloor (r-m)/2 \rfloor)$, in the central parts $(\lfloor (r-m)/2 \rfloor + 1 \le j \le \lfloor (r-m)/2 \rfloor + m)$, and in the right parts $(\lfloor (r-m)/2 \rfloor + m + 1 \le j \le r)$, respectively, where j is a column number. Then we can easily obtain

$$\begin{aligned} L &= r\left\lfloor \tfrac{r-m}{2} \right\rfloor - \tfrac{1}{2}\left\lfloor \tfrac{r-m}{2} \right\rfloor \left(\left\lfloor \tfrac{r-m}{2} \right\rfloor + 1 \right), \\ C &= 0, \\ R &= r\left\lfloor \tfrac{r-m+1}{2} \right\rfloor - \tfrac{1}{2}\left\lfloor \tfrac{r-m+1}{2} \right\rfloor \left(\left\lfloor \tfrac{r-m+1}{2} \right\rfloor + 1 \right). \end{aligned}$$

Since, for a natural number n (see [4]), $n = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n+1}{2} \rfloor$, we obtain

$$\begin{aligned} n\{B^{-1}A^{-m}\} &= L + C + R \\ &= \tfrac{1}{2}(r-m)(r+m-1) + \left\lfloor \tfrac{r-m}{2} \right\rfloor \left\lfloor \tfrac{r-m+1}{2} \right\rfloor. \end{aligned} \tag{12}$$

It is easy to observe that $n\{B^{-1}A^{-m}\}$ is a strictly decreasing function of m. On the other hand, it can be shown that

$$n\{B^{-1}A\} = \left\lfloor \tfrac{r+1}{2} \right\rfloor \left\lfloor \tfrac{r}{2} \right\rfloor + \tfrac{1}{2}\left\lfloor \tfrac{r-1}{2} \right\rfloor \left(\left\lfloor \tfrac{r-1}{2} \right\rfloor + 1 \right) < n\{B^{-1}\}$$

and

$$n\{B^{-1}A^2\} = n\{B^{-1}A^3\} = \cdots = 0.$$

Hence, we get the following relation:

$$n\{B^{-1}\} > n\{B^{-1}A\} > n\{B^{-1}A^2\} = n\{B^{-1}A^3\} = \cdots = 0.$$

Thus, (11) is obtained. This completes the proof of (b).

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AMS Classification Numbers: 11B39, 11B65, 11B83

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