THE BRAHMAGUPTA POLYNOMIALS IN TWO COMPLEX VARIABLES*

In Commemoration of Brahmagupta's Fourteenth Centenary

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1. INTRODUCTION

Some of the properties of the Brahmagupta matrix [see (1) below], and polynomials x_n and y_n in two real variables (x, y) (see § 3) have been studied in [6]; we know that the Brahmagupta polynomials contain the Fibonacci polynomials, the Pell and Pell-Lucas polynomials [2], [5], and the Morgan-Voyce polynomials [4], [7]. The convolution properties that hold for the Fibonacci polynomials and for the Pell and Pell-Lucas polynomials also hold for Brahmagupta polynomials.

In this paper we extend analytically the properties of the Brahmagupta matrix and polynomials derived in [6] from two real variables to two complex variables z and w, which belong to two distinct complex planes. We denote this space by $\mathbf{C}^2$. A typical member in $\mathbf{C}^2$ has the form $\zeta = (z, w)$. Since $\mathbf{C}$ is simply $\mathbf{R}^2$ with the additional algebraic structure, we realize that $\mathbf{C}^2$ is (topologically) $\mathbf{R}^4$ with some additional algebraic properties. We have a natural way to identify points in $\mathbf{C}^2$ with points in $\mathbf{R}^4$. This is described by the scheme:

$$\mathbf{C}^2 \ni (z, w) \leftrightarrow (x+iy, u+iv) \leftrightarrow (x, y, u, v) \in \mathbf{R}^4.$$

In particular, we measure the distance in $\mathbf{C}^2$ in the customary Euclidean fashion: if $\zeta_1 = (z_1, w_1)$ and $\zeta_2 = (z_2, w_2)$ are points in $\mathbf{C}^2$, then $|\zeta_1 - \zeta_2| = (|z_1 - z_2|^2 + |w_1 - w_2|^2)^{1/2}$.

Another interesting feature of the Brahmagupta polynomials z_n and w_n in $\mathbf{C}^2$ is that, when the polynomials are expressed in terms of real and imaginary parts with $z = x+iy$ and $w = u+iv$, the resulting polynomials x_n, y_n, u_n, v_n satisfy recurrence relations (11)-(18). The functions x_n, y_n, u_n, v_n are solutions of the second-order partial differential equations (19) and (20).

Since the calculations go through without change in the complex case, we just list some of the properties.

2. BRAHMAGUPTA MATRIX

Let B be a matrix (a Brahmagupta matrix) of the form

$$B = \begin{bmatrix} z & w \\ tw & z \end{bmatrix}, \tag{1}$$

where t is the fixed real number and z and w are complex variables; further, we shall assume that $\det B = \beta = z^2 - tw^2 \neq 0$. Using its eigenrelations, B has the following diagonal form:

$$\begin{bmatrix} z & w \\ tw & z \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{t}{2}} & -\sqrt{\frac{t}{2}} \end{bmatrix} \begin{bmatrix} z+w\sqrt{t} & 0 \\ 0 & z-w\sqrt{t} \end{bmatrix} \begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2t}} \\ \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2t}} \end{bmatrix}.$$

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Define

$$B^n = \begin{bmatrix} z & w \\ tw & z \end{bmatrix}^n = \begin{bmatrix} z_n & w_n \\ tw_n & z_n \end{bmatrix} = B_n.$$

Then the above diagonalization enables us to compute

$$B^n = \begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{t}{2}} & -\sqrt{\frac{t}{2}} \end{bmatrix} \begin{bmatrix} z+w\sqrt{t} & 0 \\ 0 & z-w\sqrt{t} \end{bmatrix}^n \begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2t}} \\ \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2t}} \end{bmatrix}. \tag{2}$$

Since $B^{n+1} = B^n B$, we have the following recurrence relations:

$$z_{n+1} = zz_n + tww_n, \qquad w_{n+1} = zw_n + wz_n, \tag{3}$$

with $z_n = z$ and $w_n = w$. From (2) we derive the following Binet forms for z_n and w_n:

$$z_n = \frac{1}{2}[(z+w\sqrt{t})^n + (z-w\sqrt{t})^n], \tag{4}$$

$$w_n = \frac{1}{2\sqrt{t}}[(z+w\sqrt{t})^n - (z-w\sqrt{t})^n], \tag{5}$$

and $z_n \pm \sqrt{t}w_n = (z \pm \sqrt{t}w)^n$. Note that if we set $z = 1/2 = w$ and $t = 5$ then $\beta = -1$; then $2w_n = F_n$ is the Fibonacci sequence, while $2z_n = L_n$ is the Lucas sequence, where $n > 0$.

Let $\xi = z + w\sqrt{t}$, $\eta = z - w\sqrt{t}$, $\xi_n = z_n + w_n\sqrt{t}$, $\eta_n = z_n - w_n\sqrt{t}$ and $\beta_n = z_n^2 - tw_n^2$, with $\eta_n = \eta$, $\xi_n = \xi$, and $\beta_n = \beta$. Then we have $\xi_n = \xi^n$, $\eta_n = \eta^n$, and $\beta_n = \beta^n$. To prove the last equality, consider $\beta^n = (z^2 - tw^2)^n = \xi^n \eta^n = \xi_n \eta_n = (z_n^2 - tw_n^2) = \beta_n$.

We also have the following property:

$$e^B = \frac{1}{4}\begin{bmatrix} e^\xi + e^\eta & \frac{1}{\sqrt{t}}(e^\xi - e^\eta) \\ \sqrt{t}(e^\xi - e^\eta) & e^\xi + e^\eta \end{bmatrix}, \quad \det e^B = e^{2z}.$$

To prove these results, set $2z_k = \xi^k + \eta^k$, $2\sqrt{t}w_k = \xi^k - \eta^k$. Since

$$e^B = \sum_{k=0}^{\infty} \frac{B^k}{k!} \quad \text{and} \quad \frac{B^k}{k!} = \frac{1}{k!}\begin{bmatrix} z_k & w_k \\ tw_k & z_k \end{bmatrix},$$

we express z_k and w_k in terms of ξ and η to obtain the desired results.

Recurrence relations (3) also imply that z_n and w_n satisfy the difference equations:

$$z_{n+1} = 2zz_n - \beta z_{n-1}, \qquad w_{n+1} = 2zw_n - \beta w_{n-1}. \tag{6}$$

Conversely, if $z_0 = 1$, $z_1 = z$, and $w_0 = 0$, and $w_1 = w$, then the solutions of the difference equations (6) are given by the Binet forms (4) and (5).

The expressions z_n and w_n can be extended to negative integers by defining $z_{-n} = z_n\beta^{-n}$ and $w_{-n} = -w_n\beta^{-n}$. Then we have

$$B^{-n} = \begin{bmatrix} z & w \\ tw & z \end{bmatrix}^n = \begin{bmatrix} z_{-n} & w_{-n} \\ tw_{-n} & z_{-n} \end{bmatrix} = B_{-n},$$

where we have used the property

$$\left(\begin{bmatrix} z & w \\ tw & z \end{bmatrix}^{-1}\right)^n = \left(\frac{1}{\beta}\begin{bmatrix} z & -w \\ -tw & z \end{bmatrix}\right)^n = \frac{1}{\beta^n}\begin{bmatrix} z_n & -w_n \\ -tw_n & z_n \end{bmatrix}.$$

All of the recurrence relations extend to the negative integers also. Notice that $B^0 = I$, where I is the identity matrix. For negative integers, z_n and w_n are rational functions of z and w.

3. THE BRAHMAGUPTA POLYNOMIALS

Using the Binet forms (4) and (5), we deduce some results: Write z_n and w_n as polynomials in z and w using the binomial expansion:

$$z_n = z^n + t\binom{n}{2}z^{n-2}w^2 + t^2\binom{n}{4}z^{n-4}w^4 + \cdots;$$

$$w_n = nz^{n-1}w + t\binom{n}{3}z^{n-3}w^3 + t^2\binom{n}{5}z^{n-5}w^5 + \cdots.$$

The first few polynomials are $z_0 = 1$, $z_1 = z$, $z_2 = z^2 + tw^2$, $z_3 = z^3 + 3tzw^2$, $z_4 = z^4 + 6tz^2w^2 + t^2w^4$, ..., $w_0 = 0$, $w_1 = w$, $w_2 = 2zw$, $w_3 = 3z^2w + tw^3$, $w_4 = 4z^3w + 4tzw^3, \ldots$. Notice that z_n and w_n are homogeneous in z and w; therefore, they are analytic (in the classical one-variable sense) in each variable separately. Also, z_n and w_n satisfy the Cauchy-Riemann equations in each variable separately: If $z_n = x_n + iy_n$, then

$$\frac{\partial x_n}{\partial x} = \frac{\partial y_n}{\partial y}, \quad \frac{\partial x_n}{\partial y} = -\frac{\partial y_n}{\partial x}$$

and

$$\frac{\partial x_n}{\partial u} = \frac{\partial y_n}{\partial v}, \quad \frac{\partial x_n}{\partial v} = -\frac{\partial y_n}{\partial u}.$$

Similar relations are satisfied by the polynomials $w_n = u_n + iv_n$.

If $t > 0$, then z_n and w_n satisfy:

$$\lim_{n\to\infty}\frac{z_n}{w_n} = \begin{cases} +\sqrt{t} & \text{if } \left|\frac{z-\sqrt{t}w}{z+\sqrt{t}w}\right| < 1, \\ -\sqrt{t} & \text{if } \left|\frac{z-\sqrt{t}w}{z+\sqrt{t}w}\right| > 1; \end{cases}$$

$$\lim_{n\to\infty}\frac{z_n}{z_{n-1}} = \lim_{n\to\infty}\frac{w_n}{w_{n-1}} = \begin{cases} z+\sqrt{t}w & \text{if } \left|\frac{z-\sqrt{t}w}{z+\sqrt{t}w}\right| < 1, \\ z-\sqrt{t}w & \text{if } \left|\frac{z-\sqrt{t}w}{z+\sqrt{t}w}\right| > 1; \end{cases}$$

$$\frac{\partial z_n}{\partial z} = \frac{\partial w_n}{\partial w} = nz_{n-1},$$

$$\frac{\partial z_n}{\partial w} = t\frac{\partial w_n}{\partial z} = ntw_{n-1}.$$

From the above relations, we infer that z_n and w_n are the polynomial solutions of the "wave equation":

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{t}\frac{\partial^2}{\partial w^2}\right)U = 0. \tag{7}$$

Since the partial differential equation (7) is linear, by the principle of superposition its general solution is

$$U(z, w) = \sum_{0}^{\infty} (A_n z_n + B_n w_n),$$

where A_n and B_n are constants.

4. RECURRENCE RELATIONS

From the Binet forms (4) and (5), we record the following obvious recurrence relations:

$$\begin{array}{llll}
\text{(i)} & z_{m+n} = z_m z_n + t w_m w_n, & \text{(vi)} & w_{m+n} + \beta^n w_{m-n} = 2 z_n w_m, \\
\text{(ii)} & w_{m+n} = z_m w_n + w_m z_n, & \text{(vii)} & z_{m+n} + \beta^n z_{m-n} = 2 t w_m w_n, \\
\text{(iii)} & \beta^n z_{m-n} = z_m z_n - t w_m w_n, & \text{(viii)} & w_{m+n} + \beta^n w_{m-n} = 2 z_m w_n, \\
\text{(iv)} & \beta^n w_{m-n} = z_n w_m - z_m w_n, & \text{(ix)} & 2(z_m^2 - z_{m+n} z_{m-n}) = \beta^{(m-n)}(\beta^n - z_{2n}), \\
\text{(v)} & z_{m+n} + \beta^n z_{m-n} = 2 z_m z_n, & \text{(x)} & z_{2m} - 2t w_{m+n} w_{m-n} = \beta^{(m-n)} z_{2n}.
\end{array} \quad (8)$$

Putting $m = n$ in (i) and (ii) above, we see that $z_{2n} = z_n^2 + t w_n^2$ and $w_{2n} = 2 z_n w_n$; these relations imply that: (a) z_{2n} is divisible by $z_n \pm i\sqrt{t} w_n$ if $t > 0$; (b) z_{2n} is divisible by $z_n \pm \sqrt{t} w_n$ if $t < 0$; (c) w_{2n} is divisible by z_n and w_n and, if r divides s, then z_{rn} and w_{rn} are divisors of w_{sn}.

Let $\sum_{k=1}^{n} = \sum$. Then, using the Binet forms, it is not difficult to see the following facts:

(i) $$\sum z_k = \frac{\beta z_n - z_{n+1} + z - \beta}{\beta - 2z + 1},$$

(ii) $$\sum w_k = \frac{\beta w_n - w_{n+1} + w}{\beta - 2z + 1},$$

(iii) $$\sum z_k^2 = \frac{\beta z_{2n} - z_{2n+2} + z_2 - \beta}{2(\beta - 2z_2 + 1)} + \frac{\beta(\beta^n - 1)}{2(\beta - 1)},$$

(iv) $$\sum w_k^2 = \frac{\beta^2 z_{2n} - z_{2n+2} + z_2 - \beta^2}{2t(\beta^2 - 2z_2 + 1)} - \frac{\beta(\beta^n - 1)}{2t(\beta - 1)},$$

(v) $$2\sum z_k z_{n+1-k} = n z_{n+1} + \frac{\beta w_n}{w},$$

(vi) $$2t\sum w_k w_{n+1-k} = n z_{n+1} - \frac{\beta w_n}{w}$$

(vii) $$2\sum z_k w_{n-k+1} = 2\sum w_k z_{n-k+1} = n w_{n+1}.$$

Now we generalize a result satisfied by the generating functions of Fibonacci (F_n) and Lucas (L_n) sequences; namely,

$$F(t) = \sum_{1}^{\infty} \frac{F_n}{n} t^n, \qquad L(t) = \sum_{1}^{\infty} L_n t^n.$$

Then $L(t) = e^{2F(t)}$ [3]. A similar result holds between z_n and w_n. Let Z and W be generating functions of z_n and w_n, respectively; that is,

$$Z=\sum_1^\infty \frac{z_n}{n}s^n, \qquad W=\sum_1^\infty w_n s^n. \tag{9}$$

Then $W(s)=swe^{2Z(s)}$. Since the proof is similar to the real case (see [6]), we omit it here.

5. SERIES SUMMATION INVOLVING RECIPROCALS OF z_n AND w_n

All the properties of infinite series summation involving x_n and y_n can be extended to the complex variables case also. Since the arithmetic goes through without any changes, we shall just list them here. For details, see [6].

1. $\sum_{k=1}^\infty \frac{1}{z_{k+1}}\left(\frac{2z}{z_{k-1}}-\frac{\beta+1}{z_k}\right)=\frac{1}{z}.$

2. $\sum_{k=r+1}^\infty \left(\frac{2z}{z_{k-1}z_{k+1}}-\frac{\beta+1}{z_{k+1}z_k}\right)=\frac{1}{z_r z_{r+1}}, \qquad \sum_{k=r+1}^\infty \left(\frac{2z}{w_{k-1}w_{k+1}}-\frac{\beta+1}{w_{k+1}w_k}\right)=\frac{1}{w_r w_{r+1}}.$

3. $\sum_{k=r+1}^\infty \frac{2zz_k}{z_{k-1}z_{k+1}}=\sum_{k=r+1}^\infty\left(\frac{1}{z_{k-1}}+\frac{\beta}{z_{k+1}}\right), \qquad \sum_{k=r+1}^\infty \frac{2zw_k}{w_{k-1}w_{k+1}}=\sum_{k=r+1}^\infty\left(\frac{1}{w_{k-1}}+\frac{\beta+1}{w_{k+1}}\right).$

4. $\sum_{k=2}^\infty \frac{1}{z_{(k+1)r}}\left(\frac{2z_r}{z_{(k-1)r}}-\frac{\beta^r+1}{z_{kr}}\right)=\frac{1}{z_r z_{2r}}, \qquad \sum_{k=2}^\infty \frac{1}{w_{(k+1)r}}\left(\frac{2z_r}{w_{(k-1)r}}-\frac{\beta^r+1}{w_{kr}}\right)=\frac{1}{w_r w_{2r}}.$

5. $\sum_{k=2}^\infty \frac{\beta^{2^{k-1}-2}}{y_{2^k}}=\frac{1}{(x+y\sqrt{t})^2}.$

6. $\sum_{n=1}^\infty \frac{\beta^{n-1}}{z_n z_{n+k}}=\frac{1}{tww_k}\left(\sum_1^k \frac{z_{n-1}}{z_n}-k(z\pm\sqrt{t}w)\right),$

where the plus sign should be taken if $|\xi/\eta|<1$ and the minus sign should be taken if $|\xi/\eta|>1$. To show item 6, we consider

$$\begin{aligned} z_{n-1}z_{n+k}-z_{n+k-1}z_n &= z_{n-1}(zz_{n+k-1}+tww_{n+k-1})-z_{n+k-1}(zz_{n-1}+tww_{n-1}) \\ &= tw(z_{n-1}w_{n+k-1}-z_{n+k-1}w_{n-1})=tw\beta^{n-1}w_k. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{n=1}^N \frac{\beta^n}{z_n z_{n+k}} &= \frac{1}{tww_k}\,\frac{z_{n-1}z_{n+k}-z_{n+k-1}z_n}{z_n z_{n+k}} \\ &= \frac{1}{tww_k}\sum_{n=1}^N\left(\frac{z_{n-1}}{z_n}-\frac{z_{n+k-1}}{z_{n+k}}\right)=\frac{1}{tww_k}\left(\sum_{n=1}^k\frac{z_{n-1}}{z_n}-\sum_{n=N+1}^{N+k}\frac{z_{n-1}}{z_n}\right). \end{aligned}$$

Now fix $k\geq 1$ and let N tend to infinity. Using the property we derived in Section 3, we obtain the required result. Similarly, we show that

$$\beta^k\sum_{n=1}^\infty \frac{\beta^{(n-1)}}{w_n w_{n+k}}=\frac{1}{ww_k}\left(\sum_1^k\frac{w_{n-1}}{w_n}-k(z\pm\sqrt{t}w)\right),$$

where the plus sign should be taken if $|\xi/\eta|<1$ and the minus sign should be taken if $|\xi/\eta|>1$.

6. CONVOLUTIONS FOR z_n AND w_n

Suppose that $a_n(z, w)$ and $b_n(z, w)$ are two homogeneous polynomial sequences in two variables z and w, where n is an integer ≥ 1. Their *first convolution sequence* is defined by

$$(a_n * b_n)^{(1)} = \sum_{j=1}^{n} a_j b_{n+1-j} = \sum_{j=1}^{n} b_j a_{n+1-j}.$$

In the above definition, we have written $a_n = a_n(z, w)$ and $b_n = b_n(z, w)$. To determine the convolutions $z_n * z_n$, $z_n * w_n$, and $w_n * w_n$, we use the matrix properties of B, namely,

$$\begin{bmatrix} z & w \\ tw & z \end{bmatrix}^{n+1} = \begin{bmatrix} z_{n+1} & w_{n+1} \\ tw_{n+1} & z_{n+1} \end{bmatrix} = B^{n+1} = B^j B^{n+1-j} = \begin{bmatrix} z_j & w_j \\ tw_j & z_j \end{bmatrix} \begin{bmatrix} z_{n+1-j} & w_{n+1-j} \\ tw_{n+1-j} & z_{n+1-j} \end{bmatrix}.$$

Let

$$B_n^{(1)} = \sum_{j=1}^{n} B_j B_{n+1-j} = \sum_{j=1}^{n} B^{n+1} = \begin{bmatrix} z_n^{(1)} & w_n^{(1)} \\ tw_n^{(1)} & z_n^{(1)} \end{bmatrix}.$$

Note that $B^n = B_n$. We prefer using the subscript notation. Since $\sum_{j=1}^{n} B_{n+1} = nB_{n+1}$, the above result can be written as

$$nB_{n+1} = \begin{bmatrix} z_n * z_n + tw_n * w_n & 2z_n * w_n \\ 2tz_n * w_n & z_n * z_n + tw_n * w_n \end{bmatrix} = \begin{bmatrix} z_n^{(1)} & w_n^{(1)} \\ tw_n^{(1)} & z_n^{(1)} \end{bmatrix} = B_n^{(1)},$$

where we have written $\sum_{j=1}^{n} = \sum$. Therefore, we have $z_n^{(1)} = nz_{n+1}$ and $w_n^{(1)} = nw_{n+1}$, or

$$2z_n * z_n = nz_{n+1} + \frac{\beta w_n}{w} \quad \text{and} \quad 2tw_n * w_n = nz_{n+1} - \frac{\beta w_n}{w},$$

from (8) parts (v) and (vi). The above result can be extended to the k^{th} convolution by defining

$$B_n^{(k)} = \sum_{j=1}^{n} B_j \left(B_{n+1-j}^{(k-1)} \right).$$

We can show that

$$B_n^{(k)} = \binom{n+k-1}{k} B_{n+k}.$$

We shall prove the result by induction on k. Since $B^{(1)} = nB_{n+1}$, the result is true for $k = 1$. Now consider

$$\begin{aligned} B_n^{(k+1)} &= \sum B_j B_{n+1-j}^{(k)} = \sum B_{n+1-j} (B_j^{(k)}) \\ &= \sum B_{n+1-j} \binom{j+k-1}{k} B_{j+k} = B_{n+k+1} \sum \binom{j+k-1}{k} = \binom{n+k}{k+1} B_{n+k+1}, \end{aligned}$$

which completes the induction. We have used the property $\sum \binom{j+k-1}{k} = \binom{n+k}{k+1}$, to derive the above result.

From the above results, we can write the following k^{th} convolutions, namely,

$$z_n^{(k)} = \binom{n+k-1}{k} z_{n+k} \quad \text{and} \quad w_n^{(k)} = \binom{n+k-1}{k} w_{n+k}. \tag{10}$$

Result (10) enables us to write the convolutions $z_n * z_n^{(k)}$, $w_n * w_n^{(k)}$, $z_n * w_n^{(k)}$, and $w_n * z_n^{(k)}$. First, we shall show that

$$2z_n * z_n^{(k)} = \binom{n+k}{k+1} z_{n+k+1} + \sum_{j=1}^{n} z_j^k z_{n-j+1} \beta^{j+k} z_{n+1-2j-k}.$$

We consider

$$\begin{aligned}
2z_n * z_n^{(k)} &= 2\sum z_j^k z_{n-j+1} \\
&= 2\sum \binom{j+k-1}{k} z_{j+k} z_{n-j+1} \\
&= 2\sum \binom{j+k-1}{k} (z_j z_k + tw_j w_k) z_{n-j+1} \\
&= 2z_k \sum \binom{j+k-1}{k} z_j z_{n-j+1} + 2tw_k \sum_{j=1}^{n} \binom{j+k-1}{k} w_j z_{n-j+1} \\
&= z_k \sum \left[\binom{j+k-1}{k} z_{n+1} + \beta^j z_{n-2j+1} \right] + w_k \sum \binom{j+k-1}{k} (w_{n+1} - \beta^j w_{n-2j+1}) \\
&= \sum \binom{j+k-1}{k} (z_k z_{n+1} + tw_k w_{n+1}) \sum \beta^j \binom{j+k-1}{k} (z_k z_{n-2j+1} - tw_k w_{n-2j+1}) \\
&= \binom{n+k}{k+1} z_{n+k+1} + \sum \binom{j+k-1}{k} \beta^{j+k} z_{n+1-2j-k}.
\end{aligned}$$

We have used (10) and (8) part (i) to derive the above result. Similarly, we can show that

$$2tw_n * w_n^{(k)} = \binom{n+k}{k+1} z_{n+k+1} - \sum \binom{j+k-1}{k} \beta^{j+k} z_{n+1-2j-k},$$

$$2z_n^{(k)} * w_n = \binom{n+k}{k+1} w_{n+k+1} - \sum \binom{j+k-1}{k} \beta^{j+k} w_{n+1-2j-k},$$

$$2z_n * w_n^{(k)} = \binom{n+k}{k+1} w_{n+k+1} - \sum \binom{j+k-1}{k} \beta^{j+k} w_{n+1-2j-k}.$$

7. THE IMPLICATIONS OF z_n AND w_n IN R^4

Let $z = x + iy$ and $w = u + iv$. Then $z_n = x_n + iy_n$, $w_n = u_n + iv_n$, and $\beta = z^2 - tw^2 = \alpha + i\gamma$, where $\alpha = x^2 - y^2 - t(u^2 - v^2)$ and $\gamma = 2(xy - tuv)$. Note that $\det B \neq 0$ implies that either $\alpha \neq 0$ or $\gamma \neq 0$. Recurrence relations (3) now become:

$$x_{n+1} = 2xx_n - 2yy_n - \alpha x_{n-1} + \gamma y_{n-1}, \tag{11}$$

$$y_{n+1} = 2yx_n + 2xy_n - \gamma x_{n-1} - \alpha y_{n-1}, \tag{12}$$

$$u_{n+1} = 2xu_n - 2yv_n - \alpha u_{n-1} + \gamma v_{n-1}, \tag{13}$$

$$v_{n+1} = 2xv_n + 2yu_n - \gamma u_{n-1} - \alpha v_{n-1}, \tag{14}$$

with $x_0 = 1,\ y_0 = 0,\ u_0 = 0,\ v_0 = 0$ and $x_1 = x,\ y_1 = y,\ u_1 = u,\ v_1 = v$. By (11)-(14), the first few polynomials are given by

$$\begin{aligned} x_2 &= x^2 - y^2 + t(u^2 - v^2), \\ y_2 &= 2(xy + tuv), \\ u_2 &= 2(xu - yv), \\ v_2 &= 2(xv + yu). \end{aligned}$$

$$\begin{aligned} x_3 &= x^3 - 3xy^2 + 3txu^2 - 3txv^2 - 6tyuv, \\ y_3 &= 3x^2y - y^3 + 6txuv + 3tyu^2 - 3tyv^2, \\ u_3 &= 3x^2u - 3y^2u - 6xyv - 3tuv^2 + tu^3, \\ v_3 &= 6xyu + 3x^2v - 3y^2v + 3tu^2v - tv^3, \ldots . \end{aligned}$$

By expressing equations (8) parts (i) and (ii) in terms of the real and imaginary components, we find that the recurrence relations transform to

$$x_{m+n} = x_n x_n - y_m y_n + t(u_m v_n - u_n v_m), \tag{15}$$

$$y_{m+n} = x_m y_n + x_n y_m + t(u_m v_n - u_n v_m), \tag{16}$$

$$u_{m+n} = x_m u_n + x_n u_m - y_m v_n - y_n v_m, \tag{17}$$

$$v_{m+n} = x_m v_n + x_n v_m + y_m u_n + y_n u_m. \tag{18}$$

To transform the partial differential equation (7) in z and w to the one in variables x, y, u, and v, we use the partial differential operators:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Then equation (7) becomes

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{1}{t}\left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial u^2}\right)\right] f_n = 0, \tag{19}$$

$$\left(\frac{\partial^2}{\partial x \partial y} - \frac{1}{t}\frac{\partial^2}{\partial u \partial v}\right) g_n = 0. \tag{20}$$

where $f_n = x_n$ or u_n and $g_n = y_n$ or v_n. By the principle of superposition, the solution of differential equations (19) and (20) are, respectively,

$$f(x, y, u, v) = \sum_0^\infty (a_n x_n + b_n u_n) \quad \text{and} \quad g(x, y, u, v) = \sum_0^\infty (c_n y_n + d_n v_n),$$

where a_n, b_n, c_n, and d_n are constants.

Now we express relation (9) in Section 4, i.e., $W(s) = swe^{2Z(s)}$, in terms of real and imaginary parts. Let $Z(s) = X(s) + iY(s)$ and $W(s) = U(s) + iV(s)$. Then (9) transforms to

$$U(s) = use^{X(s)}(u \cos Y(s) - v \sin Y(s))$$

and

$$V(s) = vse^{X(s)}(v \cos Y(s) + u \sin Y(s)).$$

Now, let us turn our attention to the convolutions in Section 6. Result (11), expressed in terms of real and imaginary components, becomes

$$x_n^{(k)} = \binom{n+k-1}{k} x_{n+k}, \qquad y_n^{(k)} = \binom{n+k-1}{k} y_{n+k},$$

$$u_n^{(k)} = \binom{n+k-1}{k} u_{n+k}, \qquad v_n^{(k)} = \binom{n+k-1}{k} v_{n+k}.$$

We have seen here some of the properties of the matrix B with complex entries; we are sure there are many more of them.

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