ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-539 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Let

$$H_m(p) = \sum_{j=1}^{m} B\left(\frac{j}{2}, p\right), m \in N,\ p > 0,$$

where

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

denotes the Betafunction. Show that for all positive reals p and all positive integers n,

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}H_{2k}(p) = 4^{n+p-1}B(n+p, n+p-1) + \frac{1}{n+p-1}. \tag{1}$$

From (1), deduce the identities

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{k}{4^k}\binom{n}{k}\binom{2k}{k} = \frac{2}{4^n}\binom{2n-2}{n-1} \tag{2}$$

and

$$\sum_{k=1}^{n}(-1)^{k-1}4^k\binom{n}{k}\Big/\binom{2k}{k} = \frac{2n}{2n-1}. \tag{3}$$

H-540 ***Proposed by Paul S. Bruckman, Highwood, IL***

Consider the sequence $U = \{u(n)\}_{n=1}^{\infty}$, where $u(n) = [n\alpha]$, its characteristic function $\delta_U(n)$, and its counting function $\pi_U(n) \equiv \sum_{k=1}^{n}\delta_U(k)$, representing the number of elements of U that are $\leq n$. Prove the following relationships:

(a) $\delta_U(n) = u(n+1) - u(n) - 1,\ n \geq 1;$

(b) $\pi_U(F_n) = F_{n-1},\ n > 1.$

H-541 ***Proposed by Stanley Rabinowitz, Westford, MA***

The simple continued fraction expansion for F_{13}^5 / F_{12}^5 is

$$11+\cfrac{1}{11+\cfrac{1}{375131+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{9+\cfrac{1}{11}}}}}}}}}}}}}}$$

This can be written more compactly using the notation $[11,11,375131,1,1,1,1,1,1,1,1,1,2,9,11]$. To be even more concise, we can write this as $[11^2,375131,1^9,2,9,11]$, where the superscript denotes the number of consecutive occurrences of the associated number in the list.

If $n>0$, prove that the simple continued fraction expansion for $(F_{10n+3}/F_{10n+2})^5$ is

$$[11^{2n}, x, 1^{10n-1}, 2, 9, 11^{2n-1}],$$

where x is an integer and find x.

SOLUTIONS

A Fibo Matrix?

H-522 ***Proposed by N. Gauthier, Royal Military College, Kingston, Ontario, Canada (Vol. 35, no. 1, February 1997)***

Let A and B be the following 2×2 matrices:

$$A=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} \quad\text{and}\quad B=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Show that, for $m\geq 1$,

$$\sum_{n=0}^{m-1} 2^n A^{2^n}(A^{2^n}+B^{2^n})^{-1}=c_{2^m}C_{2^m}-(A+B),$$

where

$$c_m=m/(F_{m+1}+F_{m-1}-2) \quad\text{and}\quad C_m=\begin{pmatrix}F_{m+1}-1 & F_m\\ F_m & F_{m-1}-1\end{pmatrix};$$

F_m is the m^{th} Fibonacci number.

Solution by Paul S. Bruckman, Highwood, IL

We begin by noting that the matrix B is the identity matrix I (as is any power of B). Let S_m denote the sum in the left member of the statement of the problem; let $W(n)=nA^n(A^n+I)^{-1}$. Note that

$$c_m=m(L_m-2)^{-1},\ c_2=2,\ C_2=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix},\ A+B=\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}.$$

Now

$$A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

a well-known result. Then $|A^n + I| = F_{n+1}F_{n-1} + F_{n+1} + F_{n-1} + 1 - (F_n)^2 = L_n + 1 + (-1)^n = L_n + 2e_n$, where e_n is the characteristic function of the even integers. Then

$$(A^n + I)^{-1} = (L_n + 2e_n)^{-1} \begin{pmatrix} F_{n-1}+1 & -F_n \\ -F_n & F_{n+1}+1 \end{pmatrix},$$

and

$$W(n) = n(L_n + 2e_n)^{-1} \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \begin{pmatrix} F_{n-1}+1 & -F_n \\ -F_n & F_{n+1}+1 \end{pmatrix}$$

$$= n(L_n + 2e_n)^{-1} \begin{pmatrix} F_{n+1}+(-1)^n & F_n \\ F_n & F_{n-1}+(-1)^n \end{pmatrix},$$

after simplification. In particular,

$$W(1) = S_1 = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

Note that

$$c_2C_2 - (A+I) = 2\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = S_1;$$

thus, the statement of the problem is valid for $m = 1$.

Let N denote the set of positive integers m for which the statement of the problem is valid. As we have just shown, $1 \in N$. Suppose that $m \in N$. Then, letting $u = 2^m$ and using the inductive hypothesis,

$$S_{m+1} = S_m + W(u) = c_uC_u - (A+I) + u(L_u+2)^{-1} \begin{pmatrix} F_{u+1}+1 & F_u \\ F_u & F_{u-1}+1 \end{pmatrix}$$

$$= u(L_u-2)^{-1} \begin{pmatrix} F_{u+1}-1 & F_u \\ F_u & F_{u-1}-1 \end{pmatrix} + u(L_u+2)^{-1} \begin{pmatrix} F_{u+1}+1 & F_u \\ F_u & F_{u-1}+1 \end{pmatrix} - (A+I)$$

$$= u\{(L_u)^2 - 4\}^{-1} \begin{pmatrix} (L_u+2)\{F_{u+1}-1\} + (L_u-2)\{F_{u+1}+1\} & 2L_uF_u \\ 2L_uF_u & (L_u+2)\{F_{u-1}-1\} + (L_u-2)\{F_{u-1}+1\} \end{pmatrix}$$
$$- (A+I)$$

$$= 2u(L_{2u}-2)^{-1} \begin{pmatrix} L_uF_{u+1}-2 & F_{2u} \\ F_{2u} & L_uF_{u-1}-2 \end{pmatrix} - (A+I)$$

$$= c_{2u} \begin{pmatrix} F_{2u+1}-1 & F_{2u} \\ F_{2u} & F_{2u-1}-1 \end{pmatrix} - (A+I) = c_{2u}C_{2u} - (A+I).$$

Comparison with the expression given in the statement of the problem shows, therefore, that $m \in N$ implies $(m+1) \in N$. This is the required inductive step, and the desired result is proven.

Also solved by H. Kappus, H.-J. Seiffert, and the proposer.

Enter!

H-523 ***Proposed by Paul S. Bruckman, Highwood, IL***
(Vol. 35, no. 1, February 1997)

Let $Z(n)$ denote the "Fibonacci entry-point" of n, i.e., $Z(n)$ is the smallest positive integer m such that $n|F_m$. Given any odd prime p, let $q=\frac{1}{2}(p-1)$; for any integer s, define $g_p(s)$ as follows:

$$g_p(s)=\sum_{k=1}^{q}\frac{s^k}{k}.$$

Prove the following assertion:

$$Z(p^2)=Z(p) \text{ iff } g_p(1)\equiv g_p(5) \pmod{p}. \quad (*)$$

Solution by H.-J. Seiffert, Berlin, Germany

We need the following results.

Proposition 1: For all positive integers n, it holds that:

(a) $2^{n-1}=\sum_{k=0}^{[n/2]}\binom{n}{2k}$;

(b) $2^{n-1}L_n=\sum_{k=0}^{[n/2]}\binom{n}{2k}5^k$.

Proof: The first equation can be found on page 4 in [1] and the second on page 69 in [3].

Proposition 2: If p is any odd prime, then $Z(p^2)=Z(p)$ if and only if $L_p\equiv 1 \pmod{p^2}$.

Proof: Since $Z(25)=25\neq 5=Z(5)$ and $L_5=11\not\equiv 1 \pmod{25}$, we do suppose that $p\neq 5$. Then (see [2], p. 386, Lemma 5), $Z(p^2)=Z(p)$ if and only if $F_{p-e}\equiv 0 \pmod{p^2}$, where $e=(5|p)$ denotes Legendre's symbol, and (see [4], p. 367, eq. (2.10)) $F_{p-e}\equiv 2e(F_p-e) \pmod{p^2}$. Our claim now easily follows from $p\neq 5$, $e\in\{-1,+1\}$, and the equations $L_p=2F_{p+1}-F_p=F_p+2F_{p-1}$. Q.E.D.

Lemma: If p is a prime, then

$$\binom{p}{j}\equiv(-1)^{j+1}\frac{p}{j} \pmod{p^2},\ j=1,2,\dots,p-1.$$

Proof: For $j=1,2,\dots,p-1$, we have

$$(-1)^j\binom{p}{j}=\frac{(-p)(1-p)\cdots(j-1-p)}{1\cdot 2\cdot\cdots\cdot(j-1)\cdot j}\equiv-\frac{p}{j} \pmod{p^2}.$$

This proves this well-known congruence. Q.E.D.

Let p be an odd prime. From Proposition 1(a) and the lemma, modulo p^2 we obtain

$$2^{p-1}=1+\sum_{k=1}^{q}\binom{p}{2k}\equiv 1-\frac{p}{2}g_p(1) \pmod{p^2}$$

or, equivalently,

$$pg_p(1) \equiv 2 - 2^p \pmod{p^2}. \tag{1}$$

Similarly, using Proposition 1(b) and the above lemma, modulo p^2 we find

$$2^{p-1}L_p = 1 + \sum_{k=1}^{q} \binom{p}{2k} 5^k \equiv 1 - \frac{p}{2} g_p(5) \pmod{p^2},$$

giving

$$pg_p(5) \equiv 2 - 2^p L_p \pmod{p^2}. \tag{2}$$

Hence, by (1) and (2), we have $g_p(1) \equiv g_p(5) \pmod{p}$ if and only if $L_p \equiv 1 \pmod{p^2}$. The desired equivalence relation now follows from Proposition 2.

Remark: In 1960, D. D. Wall posed the problem of whether there exists a prime p such that $p^2 | F_{p-e}$. It is still not known whether such a prime exists although it is known that it must exceed 10^9 (see [4], p. 366). In [2] (p. 384, Theorem 4), it was proved that if p is an odd prime such that Fermat's last theorem fails for the exponent p in the first case, then $p^2 | F_{p-e}$. Conversely, it seems that Andrew Wiles' proof of Fermat's last theorem does not imply that such primes cannot exist.

References

1. I. S. Gradsteyn & I. M. Ryzhik. *Table of Integrals, Series, and Products*. 5th ed. New York:: Academic Press, 1994.
2. Z. H. Sun & Z.-W. Sun. "Fibonacci Numbers and Fermat's Last Theorem." *Acta Arith.* **60** (1992):371-88.
3. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section*. New York: Halsted, 1989.
4. H. C. Williams. "A Note on the Fibonacci Quotient F_{p-e}/p." *Can. Math. Bull.* **25** (1982): 366-70.

Also solved by the proposer.

Z(p) ed di do da

H-524 ***Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 35, no. 1, February 1997)***

Let p be a prime with $p \equiv 1$ or $9 \pmod{20}$. It is known that $a := (p-1)/Z(p)$ is an even integer, where $Z(p)$ denotes the entry-point in the Fibonacci sequence [1]. Let $q := (p-1)/2$. Show that

(1) $(-1)^{a/2} \equiv (-5)^{q/2} \pmod{p}$ if $p \equiv 1 \pmod{20}$,

(2) $(-1)^{a/2} \equiv -(-5)^{q/2} \pmod{p}$ if $p \equiv 9 \pmod{20}$.

Reference

1. P. S. Bruckman. Problem H-515. *The Fibonacci Quarterly* **34.4** (1996):379.

Solution by Paul S. Bruckman, Highwood, IL

We will make use of the following easily verified (or well-known) results:

(a) $p | F_r$ and $F_p - e = F_r L_{r+e}$ iff $\left(\frac{-1}{p}\right) = 1$;

(b) $p | L_r$ and $F_p - e = F_{r+e} L_r$ iff $\left(\frac{-1}{p}\right) = -1$;

(c) $e = (-1)^r \left(\frac{-1}{p}\right)$;

(d) $5F_r^2 - L_{r+e}L_{r-e} = 5F_{r+e}F_{r-e} - L_r^2 = (-1)^r$;
(e) for all positive integers m and $n > 1$, $Z(m)|n$ iff $m|F_n$;
(f) $Z(p)|(p-e)$;
(g) $Z(p^2) = pZ(p)$ or $Z(p)$.

(A) Suppose $eA - B \equiv C \pmod{p}$. Then $eAp - Bp \equiv Cp \pmod{p^2}$

$$\Rightarrow e(2^{p-1}-1) - (5^q - e) \equiv p\sum_{k=1}^{q}\frac{5^{k-1}}{2k-1} \equiv \sum_{k=1}^{p-1}\frac{p}{k}\cdot\frac{1}{2}(1-(-1)^k)\cdot 5^{\frac{1}{2}(k-1)} \pmod{p^2}$$

$$\Rightarrow e\cdot 2^{p-1} \equiv \sum_{k=1}^{p}\frac{p}{k}\cdot\frac{1}{2}(1-(-1)^k)\cdot 5^{\frac{1}{2}(k-1)} \pmod{p^2}.$$

Now, if $1 \le k \le p$,

$$\binom{p}{k} = \frac{p}{k}\cdot\binom{p-1}{k-1} \equiv \frac{p}{k}\cdot\binom{-1}{k-1} \equiv \frac{p}{k}(-1)^{k-1} \pmod{p^2}.$$

Thus,

$$e\cdot 2^{p-1} \equiv \sum_{k=1}^{p}(-1)^{k-1}\cdot\frac{1}{2}(1-(-1)^k)\cdot\binom{p}{k}\cdot 5^{\frac{1}{2}(k-1)}$$

$$\equiv 5^{-\frac{1}{2}}\sum_{k=0}^{p}\binom{p}{k}\cdot\frac{1}{2}(1-(-1)^k)\cdot 5^{\frac{1}{2}k} \pmod{p^2}$$

$$\Rightarrow e\cdot 2^p \equiv 5^{-\frac{1}{2}}[(1+\sqrt{5})^p - (1-\sqrt{5})^p] \pmod{p^2} \Rightarrow F_p \equiv e \pmod{p^2}.$$

From (a) and (b), we see that $p|F_r$ and $p^2|F_rL_{r+e}$ if $\left(\frac{-1}{p}\right) = 1$, or $p|L_r$ and $p^2|F_{r+e}L_r$ if $\left(\frac{-1}{p}\right) = -1$. From (d) and (e), $\gcd(F_r, L_{r+e}) = \gcd(F_{r+e}, L_r) = 1$. Then $p^2|F_r$ if $\left(\frac{-1}{p}\right) = 1$, or $p^2|L_r$ if $\left(\frac{-1}{p}\right) = -1$. In any event, $p^2|F_{2r} = F_rL_r$. Then, from (e), $Z(p^2)|2r = p-e$. Since $p\nmid(p-e)$, it follows from (f) and (g) that $Z(p^2) = Z(p)$.

(B) The steps in (A) are reversible. Thus,

$$Z(p^2) = Z(p) \Rightarrow p^2|F_{2r} \Rightarrow p^2|(F_p - e) \Rightarrow eAp - Bp$$
$$\equiv C_p \pmod{p^2} \Rightarrow eA - B \equiv C \pmod{p}. \text{ Q.E.D.}$$

Also solved by the proposer.

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