ON A GENERALIZATION OF A CLASS OF POLYNOMIALS

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1. INTRODUCTION

In [1], R. André-Jeannin considered a class of polynomials $U_n(p, q; x)$ defined by

$$U_n(p, q; x) = (x+p)U_{n-1}(p, q; x) - qU_{n-2}(p, q; x), \quad n > 1,$$

with initial values $U_0(p, q; x) = 0$ and $U_1(p, q; x) = 1$.

Particular cases of $U_n(p, q; x)$ are: the well-known Fibonacci polynomials $F_n(x)$; the Pell polynomials $P_n(x)$ (see [4]); the Fermat polynomials of the first kind $\phi(x)$ (see [5], [3]); and the Morgan-Voyce polynomials of the second kind $B_n(x)$ (see [2]).

In this paper we shall consider the polynomials $\phi_n(p, q; x)$ defined by

$$\phi_n(p, q; x) = (x+p)\phi_{n-1}(p, q; x) - q\phi_{n-3}(p, q; x), \tag{1.0}$$

with initial values $\phi_{-1}(p, q; x) = \phi_0(p, q; x) = 0$ and $\phi_1(p, q; x) = 1$. The parameters p and q are arbitrary real numbers, $q \neq 0$.

Let us denote by α, β, and γ the complex numbers, so that they satisfy

$$\alpha + \beta + \gamma = p, \quad \alpha\beta + \alpha\gamma + \beta\gamma = 0, \quad \alpha\beta\gamma = -q. \tag{1.1}$$

The first few members of the sequence $\{\phi_n(p, q; x)\}$ are:

$$\phi_2(p, q; x) = p + x; \quad \phi_3(p, q; x) = p^2 + 2px + x^2; \quad \phi_4(p, q; x) = p^3 - q + 3p^2x + 3px^2 + x^3.$$

By induction on n, we can say that there is a sequence $\{c_{n,k}(p, q)\}_{n\geq 0,\, k\geq 0}$ of numbers, so that it holds

$$\phi_{n+1}(p, q; x) = \sum_{k\geq 0} c_{n,k}(p, q)x^k, \tag{1.2}$$

where $c_{n,k}(p, q) = 0$ for $k > n$ and $c_{n,n}(p, q) = 1$. Therefore, if we set $c_{-1,k}(p, q) = c_{-2,k}(p, q) = 0$, $k \geq 0$, then we have

$$\phi_{-1}(p, q; x) = \sum_{k\geq 0} c_{-2,k}(p, q)x^k \quad \text{and} \quad \phi_0(p, q; x) = \sum_{k\geq 0} c_{-1,k}(p, q)x^k.$$

Later on, we consider some other interesting sequences of numbers, define the polynomials $\phi_n^1(p, q; x)$ and $\phi_n^2(p, q; x)$, which are rising diagonal polynomials of $\phi_n(p, q; x)$ and $\phi_n^1(p, q; x)$, respectively, and finally, consider the generalized polynomials $\phi_n^m(x)$.

2. DETERMINATION OF THE COEFFICIENTS $c_{n,k}(p, q)$

The main purpose of this section is to determine the coefficients $c_{n,k}(p, q)$. First, for $n \geq 1$, $k \geq 1$, from (1.0), (1.1), and (1.2), we obtain

$$\begin{aligned} c_{n,k}(p, q) &= c_{n-1,k-1}(p, q) + pc_{n-1,k}(p, q) - qc_{n-3,k}(p, q) \\ &= c_{n-1,k-1}(p, q) + (\alpha+\beta)c_{n-1,k}(p, q) + \gamma(c_{n-1,k}(p, q) - \gamma(\alpha+\beta)c_{n-3,k}(p, q)). \end{aligned} \tag{2.0}$$

Therefore, we shall prove the following lemma.

Lemma 2.1: For every $k \geq 0$, we have

$$(1-pt+qt^3)^{-(k+1)} = \sum_{n\geq 0} d_{n,k}t^n, \tag{2.1}$$

where

$$d_{n,k} = \sum_{i+j+s=n} \binom{k+i}{k}\binom{k+j}{k}\binom{k+s}{k}\alpha^i\beta^j\gamma^s. \tag{2.2}$$

Proof: From (2.1), using (1.1), we get

$$\begin{aligned}(1-pt+qt^3)^{-(k+1)} &= (1-\alpha t)^{-(k+1)}(1-\beta t)^{-(k+1)}(1-\gamma t)^{-(k+1)} \\ &= \sum_{n\geq 0}\sum_{i+j+s=n} \binom{k+i}{k}\binom{k+j}{k}\binom{k+s}{k}\alpha^i\beta^j\gamma^s t^n.\end{aligned}$$

Statement (2.2) follows immediately from the last equality. □

Now we shall prove the following theorem.

Theorem 2.1: The coefficients $c_{n,k}(p,q)$ are given by

$$c_{n,k} = \sum_{i+j+s=n-k} \binom{k+i}{k}\binom{k+j}{k}\binom{k+s}{k}\alpha^i\beta^j\gamma^s. \tag{2.3}$$

Proof: First, let us define the generating function of the sequence $\phi_n(p,q;x)$ by

$$F(x,t) = \sum_{n\geq 0}\phi_{n+1}(p,q;x)t^n. \tag{2.4}$$

Then, using (1.0), we find

$$F(x,t) = (1-(p+x)t+qt^3)^{-1}. \tag{2.5}$$

Now, from (2.5) and (2.4), we deduce that

$$\frac{\partial^k F(x,t)}{\partial x^k} = \frac{k!t^k}{(1-(x+p)t+qt^3)^{k+1}} = \sum_{n\geq 0}\phi_{n+1+k}^{(k)}(p,q;x)t^{n+k},$$

since $\phi_{n+1}(p,q;x)$ is a polynomial of degree n. If we take $x=0$ in the last formula and recall that

$$c_{n+k,k}(p,q) = \frac{1}{k!}\phi_{n+1+k}^{(k)}(p,q;0),$$

then from (3), and by Taylor's formula, we get

$$(1-pt+qt^3)^{-(k+1)} = \sum_{n\geq 0} c_{n+k,k}(p,q)t^n. \tag{2.6}$$

Comparing (2.6) to (2.1) and (2.2), we see that

$$\begin{aligned}c_{n+k,k}(p,q) &= \frac{1}{k!}\phi_{n+1+k}^{(k)}(p,q;0) = d_{n,k} \\ &= \sum_{i+j+s=n}\binom{k+i}{k}\binom{k+j}{k}\binom{k+s}{k}\alpha^i\beta^j\gamma^s.\end{aligned} \tag{2.7}$$

By (2.7), we see that

$$c_{n,k}(p,q)=d_{n-k,k}=\sum_{i+j+s=n-k}\binom{k+i}{k}\binom{k+j}{k}\binom{k+s}{k}\alpha^i\beta^j\gamma^s.$$

This completes the proof of Theorem 2.1. □

Remarks:

(i) If $k=0$, then (2.3) becomes

$$c_{n,0}(p,q)=\sum_{i+j+s=n}\alpha^i\beta^j\gamma^s=\phi_{n+1}(p,q;0).$$

(ii) If $p=0$, then (2.1) becomes

$$(1+qt^3)^{-(k+1)}=\sum_{n\geq 0}(-1)^n\binom{k+n}{n}q^nt^{3n}.$$

Thus, we get

$$c_{n,n-3k}(0,q)=(-1)^k\binom{n-2k}{k}q^k,\ c_{n,n-3k-1}(0,q)=0,\ c_{n,n-3k-2}(0,q)=0,$$

for $k\leq[n/3]$. Now, from (1.2), we find that

$$\phi_{n+1}(0,q;x)=\sum_{k=0}^{[n/3]}c_{n,n-3k}(0,q)x^{n-3k}=\sum_{k=0}^{[n/3]}(-1)^k\binom{n-2k}{k}q^kx^{n-3k}. \tag{2.8}$$

We shall prove the following theorem.

Theorem 2.2: The coefficients $c_{n,k}(p,q)$ have the following form:

$$c_{n,k}(p,q)=\sum_{r=0}^{[(n-k(/3]}(-1)^r\binom{n-2r}{r}\binom{n-3r}{k}q^rp^{n-3r-k},\quad n\geq k. \tag{2.9}$$

Proof: Using (1.0), we see that $\phi_{n+1}(p,q;x)=\phi_{n+1}(0,q;x+p)$. Thus,

$$c_{n,k}(p,q)=\frac{1}{k!}\phi_{n+1}^{(k)}(p,q;0)=\frac{1}{k!}\phi_{n+1}^{(k)}(0,q;p).$$

Now, by (2.8), it follows that

$$\frac{1}{k!}\phi_{n+1}(0,q;p)=\sum_{r=0}^{[(n-k(/3]}(-1)^r\binom{n-2r}{r}\binom{n-3r}{k}q^rp^{n-3r-k}.$$

This is the desired equality (2.9). □

Corollary 2.1: From (2.9) or (2.3), we find that:

$$-\alpha-\beta-\gamma=-p;$$
$$(-\alpha)(-\beta)+(-\beta)(-\gamma)+(-\alpha)(-\gamma)=0;$$
$$(-\alpha)(-\beta)(-\gamma)=q.$$

Hence,

$$c_{n,k}(-p,-q)=(-1)^{n-k}c_{n,k}(p,q).$$

3. A PARTICULAR CASE

In this section we shall consider a particular case of the polynomials $\phi_n(p, q; x)$.

If $\alpha = \beta \neq \gamma$, then $\alpha = \beta = 2p/3$, $\gamma = -p/3$, and $27q = 4p^3$. In this case, by (2.1), we get

$$(1 - pt + qt^3)^{-(k+1)} = (1 - \alpha t)^{-2(k+1)}(1 - \gamma t)^{-(k+1)}$$
$$= \sum_{n\geq 0} \left(\sum_{i+j=n} \binom{2k+1+i}{i} \binom{k+j}{j} \alpha^i \gamma^j \right) t^n.$$

Therefore, we have

$$c_{n,k}(p, q) = (p/3)^{n-k} \sum_{i+j=n-k} (-1)^j 2^i \binom{2k+1+i}{i} \binom{k+j}{j}.$$

4. SOME INTERESTING SEQUENCES OF NUMBERS

Here we shall consider the following sequences of numbers.

(a) If we take $x = -p$, we get the sequence $\phi_n(p, q; -p) = 0$. This sequence has the following properties: $\phi_{3n}(p, q; -p) = \phi_{3n+2}(p, q; -p) = 0$ and $\phi_{3n+1}(p, q; -p) = (-1)^n q^n$. From relation (1.2), it follows that

$$\sum_{k=0}^{3n+l} (-1)^k p^k c_{3n+l,k}(p, q) = 0,$$

for $l = 1$, and

$$\sum_{k=0}^{3n} (-1)^k p^k c_{3n,k}(p, q) = (-1)^n q^n,$$

for $l = 2$.

(b) Using (1.0), for $x = 0$, we have the sequence $\{\phi_n(p, q; 0)\}$, which is defined by

$$\phi_n(p, q; 0) = p\phi_{n-1}(p, q; 0) - q\phi_{n-3}(p, q; 0),$$

for $n \geq 2$, with initial values $\phi_{-1}(p, q; 0) = \phi_0(p, q; 0) = 0$ and $\phi_1(p, q; 0) = 1$.

5. RISING DIAGONAL POLYNOMIALS

Now, we define the polynomials $\phi_n^1(p, q; x)$ and $\phi_n^2(p, q; x)$. Also, we define the polynomials $\phi_n^m(x)$. First, we shall write the polynomials $\phi_n(p, q; x)$ in tabular form (see Table 1). We define the polynomials $\phi_n^1(p, q;x)$ by

$$\phi_{n+1}^1(p, q; x) = \sum_{k=0}^{[n/2]} c_{n,k}^1(p, q) x^k = \sum_{k=0}^{[n/2]} c_{n-k,k}(p, q) x^k, \tag{5.1}$$

where $\phi_0^1(p, q; x) = 0$ and $c_{n,k}^1(p, q) = 0$ for $k > [n/2]$. Also, from Table 1, we get

$$\begin{aligned} &\phi_1^1(p, q; x) = 1, \quad \phi_2^1(p, q; x) = p, \quad \phi_3^1(p, q; x) = p^2 + x, \\ &\phi_4^1(p, q; x) = p^3 - q + 2px, \quad \phi_5^1(p, q; x) = p^4 - 2pq + 3p^2 x + x^2. \end{aligned} \tag{5.2}$$

TABLE 1

n	.	.	.	.	.	.	...
0	0	.	.	.	.	.	...
1	1	.	.	.	.	.	...
2	p	$+x$	.	.	.	.	...
3	p^2	$+2px$	$+x^2$	.	.	.	...
4	p^3-q	$+3p^2x$	$+3px^2$	$+x^3$	.	.	...
5	p^4-2pq	$+(4p^3-q)x$	$+6p^2x^2$	$+4px^3$	$+x^4$	.	...
6	p^5-3p^2q	$+(5p^4-6pq)x$	$+(10p^3-3q)x^2$	$+10p^2x^3$	$+5px^4$	$+x^5$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	...

In fact, we will prove the following theorem.

Theorem 5.1: The polynomials $\phi_n^1(p, q; x)$ satisfy the following recurrence relation:

$$\phi_n^1(p, q; x) = p\phi_{n-1}^1(p, q; x) + x\phi_{n-2}^1(p, q; x) - q\phi_{n-3}^1(p, q; x), \quad n \geq 3. \tag{5.3}$$

Proof: To prove (5.3), we will use the notations $\phi_n^1(x)$ and $c_{n,k}$ instead of $\phi_n^1(p, q; x)$ and $c_{n,k}(p, q)$, respectively, and proceed by induction on n. From (5.2), we see that statement (5.3) holds for $n = 3$. Suppose statement (5.3) is true for $n \geq 3$. Using (5.1), and by (2.0), we obtain

$$\begin{aligned}\phi_{n+1}^1(x) &= c_{n,0} + \sum_{k=1}^{[n/2]} c_{n-k,k}x^k \\ &= pc_{n-1,0} - qc_{n-3,0} + \sum_{k=1}^{[n/2]} (c_{n-1-k,k-1} + pc_{n-1-k,k} - qc_{n-3-k,k})x^k \\ &= p\sum_{k=0}^{[(n-1)/2]} c_{n-1-k,k}x^k - q\sum_{k=0}^{[(n-3)/2]} c_{n-3-k,k}x^k + x\sum_{k=0}^{[(n-2)/2]} c_{n-2-k,k}x^k,\end{aligned}$$

since the relation $c_{n,0} = pc_{n-1,0} - qc_{n-3,0}$ is valid for $n \geq 1$. Thus, statement (5.3) follows by the last equality. This completes the proof. □

Similarly, let $\phi_n^2(p, q; x)$ be the rising diagonal polynomial of $\phi_n^1(p, q; x)$, i.e.,

$$\phi_{n+1}^2(p, q; x) = \sum_{k=0}^{[n/3]} c_{n-k,k}^1(p, q)x^k.$$

Furthermore, if we denote the process

$$\phi_n^0(x) \mapsto \phi_n^1(x) \mapsto \phi_n^2(x) \mapsto \cdots \mapsto \phi_n^m(x)$$

by $\phi_n^0(x) \equiv \phi_n(p, q; x)$, then we have

$$c_{n,k}^0 = c_{n,k} \quad \text{and} \quad c_{n,k}^{m+1} = c_{n-k,k}^m. \tag{5.4}$$

From relations (5.4), we get

$$c_{n,k}^m = c_{n-k,k}^{m-1} = \cdots = c_{n-mk,k}^0.$$

Hence, for $k = 0$, we have

$$c_{n,0}^m = c_{n,0}^0 = c_{n,0}.$$

If $n = 0, 1, \ldots, m$, then $[n/(m+1)] = 0$, so we have

$$\phi_{n+1}^m(x) = c_{n,0}^m = c_{n,0}, \quad n = 0, 1, \ldots, m.$$

Also, we get

$$\phi_{n+1}^m(x) = \sum_{k=0}^{[n/(m+1)]} c_{n-mk,k} x^k, \tag{5.5}$$

where $c_{n,k}^m = 0$ for $k > [n/(m+1)]$. Therefore, we are going to prove the following theorem.

Theorem 5.2: The polynomials $\phi_n^m(x)$ satisfy the recurrence relation

$$\phi_{n+1}^m(x) = p\phi_n^m(x) - q\phi_{n-2}^m(x) + x\phi_{n-m}^m(x), \quad n \ge m \ge 2, \tag{5.6}$$

where $\phi_{-1}^m(x) = \phi_0^m(x) = 0$ and $\phi_{n+1}^m(x) = c_{n,0}^m$, $n = 0, 1, \ldots, m$.

Proof: We prove that (5.6) holds for $n \ge m \ge 2$. If $n = m$, then

$$\begin{aligned}\phi_{m+1}^m(x) &= c_{m,0} = pc_{m-1,0} - qc_{m-3,0}\\ &= p\phi_m^m(x) - q\phi_{m-2}^m(x) + x\phi_0^m(x) \quad (\phi_0(p, q; x) = 0).\end{aligned}$$

Assume now that $n \ge m+1$, then, by (2.0), we have

$$\begin{aligned}\phi_{n+1}^m(x) &= \sum_{k=0}^{[n/(m+1)]} c_{n-mk,k} x^k = c_{n,0} + \sum_{k=1}^{[n/(m+1)]} c_{n-mk,k} x^k\\
&= pc_{n-1,0} - qc_{n-3,0} + \sum_{k=1}^{[n/(m+1)]} (pc_{n-1-mk,k} - qc_{n-mk-3,k} + c_{n-mk-1,k-1}) x^k \quad (n - mk \ge 1)\\
&= p\sum_{k=0}^{[n/(m+1)]} c_{n-1-mk,k} x^k - q\sum_{k=0}^{[n/(m+1)]} c_{n-3-mk,k} x^k + x\sum_{k=0}^{[n/(m+1)]} c_{n-1-mk,k-1} x^{k-1}\\
&= p\sum_{k=0}^{[(n-1)/(m+1)]} c_{n-1-mk,k} x^k - q\sum_{k=0}^{[(n-3)/(m+1)]} c_{n-3-mk,k} x^k + x\sum_{k=0}^{[(n-1-m)/(m+1)]} c_{n-m-mk-1,k} x^k\\
&= p\phi_n^m(x) - q\phi_{n-2}^m(x) + x\phi_{n-m}^m(x). \quad \square\end{aligned}$$

Corollary 5.1: The coefficients $c_{n,k}^m$ satisfy the following relation,

$$c_{n,k}^m = pc_{n-1,k}^m - qc_{n-3,k}^m + c_{n-1-m,k-1}^m, \quad m \ge 0, n \ge 2, n \ge m, k \ge 1,$$

where $c_{n,k}^m = c_{n,k}^m(p, q)$.

Corollary 5.2: For $m = 2$, from (5.6), we have

$$\phi_n^2(x) = p\phi_{n-1}^2(x) + (x - q)\phi_{n-2}^2(x), \quad n \ge 2, \tag{5.7}$$

with $\phi_0^2(x) = 0$, $\phi_{n+1}^2(x) = c_{n,0}^1 = c_{n,0}$, $n = 0, 1$.

Remark: For every $n \ge 1$, we have

$$\phi_n^2(p, q; x) = \phi_n(p, x - q; 0). \tag{5.8}$$

Proof: By (1.0), the sequence $\{\phi_n(p, x-q; 0)\}$ satisfies relation (5.7) with $\phi_0(p, q-x; 0) = 0$, $\phi_1(p, q-x; 0) = 1$, $\phi_2(p, q-x; 0) = p$. From this and (5.7), we see that (5.8) holds for $n = 1$ and $n = 2$. If (5.8) holds for $n \le m$, then for $n = m+1$ we get

$$\begin{aligned}\phi^2_{m+1}(p, q; x) &= p\phi^2_m(p, q; x) - (q-x)\phi^2_{m-2}(p, q; x)\\ &= p\phi_m(p, q-x; 0) - (q-x)\phi_{m-2}(p, q-x; 0) = \phi_{m+1}(p, q-x; 0).\end{aligned}$$

Using induction on n, we conclude that relation (5.8) holds for every $n \ge 1$. By (5.8), and from (2.9) with $k = 0$, we get

$$\phi^2_{n+1}(p, q; x) = \sum_{r=0}^{[n/3]} \binom{n-2r}{r} (x-q)^r p^{n-3r}. \tag{5.9}$$

Special Cases

For $x = q$, by (5.9), we have

$$\sum_{k=0}^{[n/3]} q^k c^1_{n-k,k}(p, q) = p^n.$$

For $p = 2$ and $q = 1$, the last equality becomes

$$\sum_{k=0}^{[n/3]} c^1_{n-k,k}(2, 1) = 2^n.$$

For $p = 0$, the polynomials $\phi^2_{n+1}(p, q; x)$ have the following representations:

$$\phi^2_{n+1}(0, q; x) = (x-q)^s$$

for $n = 3s$, and

$$\phi^2_{n+1}(0, q; x) = 0$$

for $n = 3s+1$ and for $n = 3s+2$.

6. GENERALIZATION

If we consider the general recurrence relation

$$U_n(x) = (x+p)U_{n-1}(x) - qU_{n-2}(x) + rU_{n-3}(x), \quad n \ge 3,$$

we find that

$$U_{n+1}(x) = \sum_{k=0}^{n} c_{n,k}(p, q, r)x^k,$$

where

$$\sum_{n\ge 0} c_{n+k,k}(p, q, r)t^n = (1 - pt + qt^2 - rt^3)^{-(k+1)}.$$

In this case, we have $\alpha + \beta + \gamma = p$, $\alpha\beta + \alpha\gamma + \beta\gamma = q$, and $\alpha\beta\gamma = r$. Particularly, if $\alpha = \beta = \gamma = p/3$, then $q = p^2/3$ and $r = p^3/27$. So we get

$$\sum_{n\ge 0} c_{n+k,k}(p, q, r)t^n = (1-\alpha t)^{-3(k+1)} = \sum_{n\ge 0} \binom{3k+2+n}{3k+2} (p/3)^n t^n;$$

hence,

$$c_{n,k}(p,q,r) = \binom{2k+2+n}{3k+2}(p/3)^{n-k}.$$

Thus, we can define $B_n^1(x)$, i.e., a generalization of Morgan-Voyce polynomials, by setting $\alpha = \beta = \gamma = 1$ (i.e., $p = 3, q = 3, r = 1$),

$$B_n^1(x) = \sum_{k=0}^{n} \binom{n+2k+2}{3k+2} x^k.$$

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