THE FACTORIZATION OF $x^5 \pm x^a + n$

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1. INTRODUCTION

Rabinowitz [5] has determined all integers n for which $x^5 \pm x + n$ factors as a product of an irreducible quadratic and an irreducible cubic with integral coefficients. Using the properties of Fibonacci numbers, he showed that, in fact, there are only ten such integers n.

Theorem (Rabinowitz [5]): The only integral n for which $x^5 + x + n$ factors into the product of an irreducible quadratic and an irreducible cubic are $n = \pm 1$ and $n = \pm 6$. The factorizations are

$$x^5 + x \pm 1 = (x^2 \pm x \pm 1)(x^3 \mp x^2 \pm 1),$$
$$x^5 + x \pm 6 = (x^2 \pm x + 2)(x^3 \mp x^2 - x \pm 3).$$

The only integral n for which $x^5 - x + n$ factors into the product of an irreducible quadratic and an irreducible cubic are $n = \pm 15$, $n = \pm 22{,}440$, and $n = \pm 2{,}759{,}640$. The factorizations are

$$x^5 - x \pm 15 = (x^2 \pm x + 3)(x^3 \mp x^2 - 2x \pm 5),$$
$$x^5 - x \pm 22440 = (x^2 \mp 12x + 55)(x^3 \pm 12x^2 + 89x \pm 408),$$
$$x^5 - x \pm 2759640 = (x^2 \pm 12x + 377)(x^3 \mp 12x^2 - 233x \pm 7320).$$

In this paper we investigate the corresponding question for the quintics $x^5 \pm x^a + n$, where $a = 2, 3$, and 4. We show that for $a = 2, 3$ there are only finitely many n for which $x^5 \pm x^a + n$ factors as a product of an irreducible quadratic and an irreducible cubic, whereas, for $a = 4$, rather surprisingly we show that there are infinitely many such n, which can be parameterized using the Fibonacci numbers. Our treatment of the polynomials $x^5 \pm x^a + n$ makes use of the following three results about Fibonacci numbers.

Theorem (Cohn [1], [2]): The only Fibonacci numbers F_k $(k \geq 0)$ that are perfect squares are $F_0 = 0^2$, $F_1 = F_2 = 1^2$, and $F_{12} = 12^2$.

Theorem (London and Finkelstein [3]): The only Fibonacci numbers F_k $(k \geq 0)$ that are perfect cubes are $F_0 = 0^3$, $F_1 = F_2 = 1^3$, and $F_6 = 2^3$.

Theorem (Wasteels [7], May [4]): If x and y are nonzero integers such that $x^2 - xy - y^2 = \varepsilon$, where $\varepsilon = \pm 1$, then there exists a positive integer k such that

$$x = F_{k+1}, \quad y = F_k, \quad \varepsilon = (-1)^k, \quad \text{if } x > 0, y > 0,$$
$$x = F_k, \quad y = -F_{k+1}, \quad \varepsilon = (-1)^{k+1}, \quad \text{if } x > 0, y < 0,$$

$$x = -F_k, \quad y = F_{k+1}, \quad \varepsilon = (-1)^{k+1}, \quad \text{if } x < 0, y > 0,$$
$$x = -F_{k+1}, \quad y = -F_k, \quad \varepsilon = (-1)^k, \quad \text{if } x < 0, y < 0.$$

We remark that the above formulation corrects, and makes more precise, May's extension of Wasteels' theorem. To see that May's result is not correct, take $x = 13$ and $y = -8$ in part (3) of her theorem. Clearly

$$y^2 - xy - x^2 + 1 = (-8)^2 - 13(-8) - 13^2 + 1 = 64 + 104 - 169 + 1 = 0,$$

but there does not exist an integer n such that $13 = F_{n-1}$, $-8 = -F_n$, *or* $13 = -F_{n-1}$, $-8 = F_n$, since $F_{-7} = 13$, $F_{-6} = -8$, $F_6 = 8$, and $F_7 = 13$.

We prove the following results.

Theorem 1: The only integers n for which $x^5 + x^2 + n$ factors into the product of an irreducible quadratic and an irreducible cubic are $n = -90, -4, 18$, and 11466. The factorizations are

$$x^5 + x^2 - 90 = (x^2 + 4x + 6)(x^3 - 4x^2 + 10x - 15),$$
$$x^5 + x^2 - 4 = (x^2 - x + 2)(x^3 + x^2 - x - 2),$$
$$x^5 + x^2 + 18 = (x^2 + x + 3)(x^3 - x^2 - 2x + 6),$$
$$x^5 + x^2 + 11466 = (x^2 + 4x + 42)(x^3 - 4x^2 - 26x + 273).$$

The only integers n for which $x^5 - x^2 + n$ factors into the product of an irreducible quadratic and an irreducible cubic are $n = -11466, -18, 4$, and 90. The factorizations are

$$x^5 - x^2 - 11466 = (x^2 - 4x + 42)(x^3 + 4x^2 - 26x - 273),$$
$$x^5 - x^2 - 18 = (x^2 - x + 3)(x^3 + x^2 - 2x - 6),$$
$$x^5 - x^2 + 4 = (x^2 + x + 2)(x^3 - x^2 - x + 2),$$
$$x^5 - x^2 + 90 = (x^2 - 4x + 6)(x^3 + 4x^2 + 10x + 15).$$

Theorem 2: The only integers n for which $x^5 - x^3 + n$ factors into the product of an irreducible quadratic and an irreducible cubic are $n = \pm 8$. The factorizations are

$$x^5 - x^3 \pm 8 = (x^2 \pm x + 2)(x^3 \mp x^2 - 2x \pm 4).$$

There are no integers n for which $x^5 + x^3 + n$ factors into the product of an irreducible quadratic and an irreducible cubic.

Theorem 3: Apart from the factorizations

$$x^5 + x^4 + 1 = (x^2 + x + 1)(x^3 - x + 1),$$
$$x^5 - x^4 - 1 = (x^2 - x + 1)(x^3 - x - 1),$$

all factorizations of $x^5 \pm x^4 + n$ as a product of an irreducible quadratic and an irreducible cubic with n integral are given by

$$\begin{aligned} x^5 + \theta(-1)^k x^4 + \theta F_{k-1}^2 F_{k+1}^4 F_{k+2}^4 &= (x^2 + \theta F_{k-1} F_{k+2} x + F_{k-1} F_{k+1} F_{k+2}^2) \\ &\quad \times (x^3 - \theta F_k F_{k+1} x^2 - F_{k-1} F_{k+1}^2 F_{k+2} x + \theta F_{k-1} F_{k+1}^2 F_{k+2}^2) \end{aligned} \tag{1.1}$$

and

$$x^5+\theta(-1)^k x^4-\theta F_{k-1}^4 F_k^4 F_{k+2}^2=(x^2+\theta F_{k-1}F_{k+2}x+F_{k-1}^2 F_k F_{k+2}) \times (x^3-\theta F_k F_{k+1}x^2+F_{k-1}F_k^2 F_{k+2}x-\theta F_{k-1}^2 F_k^3 F_{k+2}), \tag{1.2}$$

where $\theta=\pm 1$ and k is an integer with $k \geq 2$ and F_k denotes the k^{th} Fibonacci number.

Taking $k=2$ and 3 in Theorem 3, we obtain the factorizations

$$\begin{aligned}
x^5 \pm x^4 \pm 1296 &= (x^2 \pm 3x+18)(x^3 \mp 2x^2-12x \pm 72),\\
x^5 \pm x^4 \mp 9 &= (x^2 \pm 3x+3)(x^3 \mp 2x^2+3x \mp 3),\\
x^5 \pm x^4 \mp 50625 &= (x^2 \mp 5x+75)(x^3 \pm 6x^2-45x \mp 675),\\
x^5 \pm x^4 \pm 400 &= (x^2 \mp 5x+10)(x^3 \pm 6x^2+20x \pm 40).
\end{aligned}$$

2. FACTORIZATION OF $x^5 \pm x^2 + n$

Let m and n be integers with $n \neq 0$. Suppose that

$$x^5+mx^2+n=(x^2+ax+b)(x^3+cx^2+dx+e), \tag{2.1}$$

where a, b, c, d, and e are integers. Then, equating coefficients in (2.1), we obtain

$$be=n, \tag{2.2}$$

$$ae+bd=0, \tag{2.3}$$

$$ad+bc+e=m, \tag{2.4}$$

$$b+ac+d=0, \tag{2.5}$$

$$a+c=0. \tag{2.6}$$

From (2.2), as $n \neq 0$, we deduce that

$$b \neq 0, \quad e \neq 0. \tag{2.7}$$

We show next that $a \neq 0$. Suppose, on the contrary, that $a=0$. From (2.3) we see that $bd=0$. Hence, from (2.7), we have $d=0$. Then, from (2.5), we deduce that $b=0$, contradicting (2.7). Hence, we must have

$$a \neq 0. \tag{2.8}$$

Next, we show that $a^2-2b \neq 0$. For, if $a^2-2b=0$, then, from (2.3), (2.5), and (2.6), we deduce that

$$b=a^2/2, \quad c=-a, \quad d=a^2/2, \quad e=-a^3/4. \tag{2.9}$$

Then, from (2.2), (2.4), and (2.9), we have

$$m=-a^3/4, \quad n=-a^5/8. \tag{2.10}$$

From (2.1), (2.9), and (2.10), we obtain the factorization

$$x^5-\frac{a^3}{4}x^2-\frac{a^5}{8}=\left(x^2+ax+\frac{a^2}{2}\right)\left(x^3-ax^2+\frac{a^2}{2}x-\frac{a^3}{4}\right). \tag{2.11}$$

As $-a^3/4 \neq \pm 1$, this factorization is not of the required type. Hence, we may suppose that

$$a^2-2b \neq 0. \tag{2.12}$$

Equations (2.3), (2.4), and (2.5) can be written as three linear equations in the three unknowns c, d, and e:

$$\begin{cases} bc+ad+e = m, \\ ac+d = -b, \\ bd+ae = 0. \end{cases} \tag{2.13}$$

Solving the system (2.13) for c, d, and e, we have

$$c = \frac{-am+b^2-a^2b}{a^3-2ab}, \quad d = \frac{a^2m+ab^2}{a^3-2ab}, \quad e = \frac{-abm-b^3}{a^3-2ab}. \tag{2.14}$$

Putting these values into (2.6), we obtain

$$a^4-3a^2b+b^2 = am. \tag{2.15}$$

Now let

$$a = a_1a_2^2, \tag{2.16}$$

where a_1 is a squarefree integer and a_2 is a positive integer. Then (2.15) becomes

$$a_1^4a_2^8-3a_1^2a_2^4b+b^2 = a_1a_2^2m. \tag{2.17}$$

From (2.17), we see that $a_1a_2^2 \mid b^2$, so that $a_1a_2 \mid b$, say,

$$b = a_1a_2r, \tag{2.18}$$

where r is a nonzero integer. From (2.17) and (2.18), we deduce that

$$a_1^3a_2^6-3a_1^2a_2^3r+a_1r^2 = m. \tag{2.19}$$

We now suppose that $m = \pm 1$. From (2.19), we see that $a_1 = \pm 1$. Hence, $a_1^2 = 1$ and (2.19) gives

$$a_2^6-3a_1a_2^3r+r^2 = a_1m. \tag{2.20}$$

We define integers s (> 0) and t by

$$s = a_2^3, \quad t = r-\tfrac{1}{2}(3a_1-1)s. \tag{2.21}$$

From (2.20) and (2.21), we obtain

$$t^2-st-s^2 = a_1m. \tag{2.22}$$

First, we deal with the possibility $t = 0$. If $a_1 = 1$, then, from (2.21), we deduce that $r = s$ and, from (2.22), that $-s^2 = m$. Hence, $m = -1$ and $r = s = \pm 1$. Then, by (2.21), we have $a_2 = s = \pm 1$. Hence, by (2.16) and (2.18), we have $a = 1$ and $b = 1$. Then, from (2.14), we get $e = 0$, contradicting (2.7). If $a_1 = -1$, then, from (2.21), we deduce that $r = -2s$ and, from (2.22), that $s^2 = m$. Hence, $m = 1$, $s = \pm 1$, and $r = \mp 2$. Then, by (2.21), we have $a_2 = s = \pm 1$. Next, by (2.16) and (2.18), we have $a = -1$ and $b = 2$. Then, from (2.2) and (2.14), we get $c = 1$, $d = -1$, $e = -2$, $n = -4$, and (2.1) becomes

$$x^5+x^2-4 = (x^2-x+2)(x^3+x^2-x-2),$$

which is one of the factorizations listed in Theorem 1.

Now we turn to the case $t \neq 0$. As $t \neq 0$ and $s > 0$, by the theorem of Wasteels and May, there is a positive integer k such that $s = F_k$. Thus, by (2.21), we have $F_k = a_2^3$. Appealing to the theorem of London and Finkelstein, we deduce that $s = F_k = a_2^3 = 1^3$ *or* 2^3, so that $a_2 = 1$ *or* 2. We have eight cases to consider according as $a_1 = 1$ *or* -1, $a_2 = 1$ *or* 2, $m = 1$ *or* -1. In each case, we determine a from (2.16). Then we determine the possible values of r (if any) from the quadratic equation (2.20). Next, we determine b from (2.18). Then the values of c, d, and e are determined from $c = -a$, $d = -b - ac$, and $e = -bd/a$. Finally, n is determined using $n = be$. We obtain the following table:

a_1	a_2	m	a	r	b	c	d	e	n
1	2	1	4	3	6	−4	10	−15	−90
				21	42	−4	−26	273	11466
1	2	−1	4	(none)					
1	1	1	1	0	0	(inadmissible	as	$b \neq 0$)	
				3	3	−1	−2	6	18
1	1	−1	1	1	1	−1	0	0	(inadmissible as $e \neq 0$)
				2	2	−1	−1	2	4
−1	2	1	-4	(none)					
−1	2	−1	−4	−3	6	4	10	15	90
				−21	42	4	−26	−273	−11466
−1	1	1	−1	−1	1	1	0	0	(inadmissible as $e \neq 0$)
				−2	2	1	−1	−2	−4
−1	1	−1	−1	0	0	(inadmissible	as	$b \neq 0$)	
				−3	3	1	−2	−6	−18

These give the eight factorizations listed in the statement of Theorem 1. It is easy to check in each case that the quadratic and cubic factors are irreducible.

3. FACTORIZATION OF $x^5 \pm x^3 + n$

Let m and n be integers with $n \neq 0$. Suppose that

$$x^5 + mx^3 + n = (x^2 + ax + b)(x^3 + cx^2 + dx + e), \tag{3.1}$$

where a, b, c, d, and e are integers. Equating coefficients in (3.1), we obtain

$$be = n, \tag{3.2}$$
$$ae + bd = 0, \tag{3.3}$$
$$ad + bc + e = 0, \tag{3.4}$$
$$b + ac + d = m, \tag{3.5}$$
$$a + c = 0. \tag{3.6}$$

From (3.6), we obtain

$$c = -a. \tag{3.7}$$

As $n \neq 0$, we see from (3.2) that

$$b \neq 0, \quad e \neq 0. \tag{3.8}$$

Suppose that $a = 0$. From (3.7), we have $c = 0$. Then, from (3.4), we deduce that $e = 0$, contradicting (3.8). Hence, we have

$$a \neq 0. \tag{3.9}$$

Suppose next that $b = a^2$. Then, from (3.5) and (3.7), we deduce that $d = m$. Then, from (3.4), we obtain $e = a^3 - am$. Next, (3.3) gives $a = 0$, contradicting (3.9). Thus, we have

$$b \neq a^2. \tag{3.10}$$

Using (3.7) in (3.4), we obtain

$$ad + e = ab. \tag{3.11}$$

Solving (3.3) and (3.11) for d and e, we find that

$$d = \frac{-a^2 b}{b - a^2}, \quad e = \frac{ab^2}{b - a^2}. \tag{3.12}$$

From (3.2) and (3.5), we deduce that

$$m = \frac{a^4 - 3a^2 b + b^2}{b - a^2}, \quad n = \frac{ab^3}{b - a^2}. \tag{3.13}$$

We define the nonzero integer h by

$$h = b - a^2. \tag{3.14}$$

Then, from (3.7), (3.12), (3.13), and (3.14), we obtain

$$\begin{aligned} b &= a^2 + h, & e &= \frac{a^5}{h} + 2a^3 + ah, \\ c &= -a, & m &= -\frac{a^4}{h} - a^2 + h, \\ d &= -\frac{a^4}{h} - a^2, & n &= \frac{a^7}{h} + 3a^5 + 3a^3 h + ah^2. \end{aligned}$$

These values of b, c, d, e, m, and n satisfy the equations (3.2)-(3.6). The equation for m can be rewritten as $h^2 - (a^2 + m)h - a^4 = 0$. Solving this quadratic equation for h, we obtain

$$h = \tfrac{1}{2}\left(a^2 + m + \varepsilon\sqrt{(a^2 + m)^2 + 4a^4}\right), \tag{3.15}$$

where $\varepsilon = \pm 1$. Relation (3.15) shows that $\sqrt{(a^2 + m)^2 + 4a^4}$ is an integer, namely, $\varepsilon(2h - a^2 - m)$. Hence, there is an integer w such that

$$(a^2 + m)^2 + (2a^2)^2 = w^2. \tag{3.16}$$

From (3.9) and (3.16), we see that $w \neq 0$. As $\{a^2 + m, 2a^2, w\}$ is a Pythagorean triple, there exist integers r, s, and t with $\gcd(r, s) = 1$ such that

$$a^2 + m = 2rst, \ 2a^2 = (r^2 - s^2)t, \ w = (r^2 + s^2)t \tag{3.17}$$

or

$$a^2 + m = (r^2 - s^2)t, \ 2a^2 = 2rst, \ w = (r^2 + s^2)t. \tag{3.18}$$

We assume now that $m = \pm 1$.

If (3.17) holds, then $(r^2 - 4rs - s^2)t = 2a^2 - 2(a^2 + m) = -2m = \pm 2$, so that $t = \pm 1$ or $t = \pm 2$. If $t = \pm 1$, then $r^2 - 4rs - s^2 = \pm 2$, so that $r^2 - s^2 \equiv 2 \pmod 4$, which is impossible, since $r^2 - s^2 \equiv$ 0, 1,or 3 (mod 4). Hence, $t = \pm 2$ and

$$r^2 - 4rs - s^2 = \pm 1. \tag{3.19}$$

From (3.19), we see that $r + s$ and $r - s$ are both odd integers so that, in particular, we have $r + s \neq 0$ and $r - s \neq 0$. Moreover, from (3.19), we have $(r + s)^2 - (r + s)(r - s) - (r - s)^2 = \pm 1$.

Therefore, by the theorem of Wasteels and May, there are positive integers k and l such that $|r+s| = F_k$ and $|r-s| = F_l$. Now, by (3.17), we have

$$a^2 = |r+s||r-s|. \tag{3.20}$$

As $|r+s|$ and $|r-s|$ are both odd, and $\gcd(r, s) = 1$, we have

$$\gcd(|r+s|, |r-s|) = 1. \tag{3.21}$$

From (3.20) and (3.21), we deduce that each of $|r+s|$ and $|r-s|$ is a perfect square. Hence, F_k and F_l are both perfect squares so, by Cohn's theorem, we have $|r+s| = F_k = 1$ or 144, $|r-s| = F_l = 1$ or 144. However, $|r+s|$ and $|r-s|$ are both odd, so $|r+s| = 1$ and $|r-s| = 1$. Therefore, $(r, s) = (\pm 1, 0)$ or $(0, \pm 1)$. Hence, by (3.17), we have $a^2 + m = 2rst = 0$, and, as $m = \pm 1$, we have $m = -1$, $a = \theta$, where $\theta = \pm 1$. From (3.15), we deduce that $h = \varepsilon$; thus, $a = \theta$, $b = 1+\varepsilon$, $c = -\theta$, $d = -(1+\varepsilon)$, $e = 2\theta(1+\varepsilon)$, $m = -1$, and $n = 4\theta(1+\varepsilon)$. Since $b \neq 0$, we must have $\varepsilon = 1$. Thus, $a = \theta, b = 2, c = -\theta, d = -2, e = 4\theta, m = -1, n = 8\theta$, which gives the factorization

$$x^5 - x^3 + 8\theta = (x^2 + \theta x + 2)(x^3 - \theta x^2 - 2x + 4\theta), \ \theta = \pm 1.$$

If (3.18) holds, then

$$(r^2 - rs - s^2)t = (r^2 - s^2)t - rst = (a^2 + m) - a^2 = m,$$

so that $t = \pm 1$, $r^2 - rs - s^2 = mt$. If r or $s = 0$, then, by (3.18), we have $a = 0$, contradicting (3.9). Hence, $r \neq 0$ and $s \neq 0$. Then, by the theorem of Wasteels and May, we have $|r| = F_k$, $|s| = F_l$, for positive integers k and l. Now

$$a^2 = rst = |r||s|, \ \gcd(|r|, |s|) = 1,$$

so each of $|r|$ and $|s|$ is a perfect square. Thus, both F_k and F_l are perfect squares. Hence, by Cohn's theorem, we have $|r| = F_k = 1$ or 144 and $|s| = F_l = 1$ or 144. Therefore, $r = \pm 1, \pm 144$ and $s = \pm 1, \pm 144$.

From $r^2 - rs - s^2 = mt$, we deduce that

(α) $r = 1, \ s = 1, \ mt = -1$, or
(β) $r = 1, \ s = -1, \ mt = 1$, or
(γ) $r = -1, \ s = 1, \ mt = 1$, or
(δ) $r = -1, \ s = -1, \ mt = -1$.

Then, from $a^2 = rst$ we deduce that

(α) $t = 1, \ m = -1, \ a = \theta$,
(β) $t = -1, \ m = -1, \ a = \theta$,
(γ) $t = -1, \ m = -1, \ a = \theta$,
(δ) $t = 1, \ m = -1, \ a = \theta$,

where $\theta = \pm 1$. In all four cases, $a^2 + m = 0$ so that, by (3.15), $h = \varepsilon$. Thus, $b = a^2 + h = 1 + \varepsilon$. But $b \neq 0$, so $\varepsilon \neq -1$, that is, $\varepsilon = 1$. Hence, $a = \theta$, $b = 2$, $c = -\theta$, $d = -2$, $e = 4\theta$, $m = -1$, $n = 8\theta$, which gives the same factorization as before. Since $x^2 + \theta x + 2$ and $x^3 - \theta x^2 - 2x + 4\theta$ are both irreducible, this completes the proof of Theorem 2.

4. FACTORIZATION OF $x^5 \pm x^4 + n$

Let m and n be integers with $n \neq 0$. Suppose that

$$x^5 + mx^4 + n = (x^2 + ax + b)(x^3 + cx^2 + dx + e), \tag{4.1}$$

where a, b, c, d, and e are integers. Equating coefficients in (4.1), we obtain

$$be = n, \tag{4.2}$$
$$ae + bd = 0, \tag{4.3}$$
$$ad + bc + e = 0, \tag{4.4}$$
$$b + ac + d = 0, \tag{4.5}$$
$$a + c = m. \tag{4.6}$$

As $n \neq 0$ we have, from (4.2),

$$b \neq 0, \quad e \neq 0. \tag{4.7}$$

We show next that $a \neq 0$. Suppose $a = 0$. Then, by (4.3) and (4.7), we have $d = 0$. From (4.5), we deduce that $b = 0$, contradicting (4.7). Hence,

$$a \neq 0. \tag{4.8}$$

Suppose next that $b = a^2/2$. Then, from (4.3) and (4.8) we obtain $e = -ad/2$. Next, from (4.4) and (4.8), we deduce that $d = -ac$. Then (4.5) gives $b = 0$, contradicting (4.7). Hence,

$$b \neq a^2/2. \tag{4.9}$$

If $a = m$ then, from (4.6), we have $c = 0$. Then (4.5) gives $d = -b$. Next, (4.4) gives $e = bm$. Now (4.3) and (4.7) give $b = m^2$, so that $e = m^3$ and $d = -m^2$. Finally, from (4.2), we obtain $n = m^5$. Thus, (4.1) becomes

$$x^5 + mx^4 + m^5 = (x^2 + mx + m^2)(x^3 - m^2 x + m^3).$$

With $m = \pm 1$ we have

$$x^5 + mx^4 + m = (x^2 + mx + 1)(x^3 - x + m).$$

It is easy to check that $x^2 + mx + 1$ and $x^3 - x + m$ are irreducible for $m = \pm 1$.

Thus, we may suppose from now on that $a \neq m$. Replacing x by $-x$ in (4.1), we obtain the factorization

$$x^5 - mx^4 - n = (x^2 - ax + b)(x^3 - cx^2 + dx - e).$$

Thus, in view of (4.8), we may suppose without loss of generality that $a > 0$. Solving (4.3), (4.4), and (4.5) for c, d, and e, we obtain

$$c = \frac{-b(a^2 - b)}{a(a^2 - 2b)}, \tag{4.10}$$

$$d = \frac{b^2}{a^2 - 2b}, \tag{4.11}$$

$$e = \frac{-b^3}{a(a^2 - 2b)}. \tag{4.12}$$

Then, from (4.6) and (4.2), we deduce that

$$m = \frac{a^4 - 3a^2b + b^2}{a(a^2 - 2b)}, \quad n = \frac{-b^4}{a(a^2 - 2b)}. \tag{4.13}$$

Assume now that $m = \pm 1$. Writing the equation for m in (4.13) as a quadratic equation in b, we have

$$b^2 + a(2m - 3a)b + a^3(a - m) = 0.$$

Solving for b, we find that

$$b = \frac{a}{2}(3a - 2m + \varepsilon\sqrt{5a^2 - 8ma + 4}), \tag{4.14}$$

where $\varepsilon = \pm 1$. The equation (4.14) shows that $z = +\sqrt{5a^2 - 8ma + 4}$ is a nonnegative rational number. As a and m are integers, z must be a nonnegative integer such that $5a^2 - 8ma + 4 = z^2$, that is,

$$a^2 + (2a - 2m)^2 = z^2. \tag{4.15}$$

As $a \neq 0$ and $a \neq m$, we have $z \geq 2$, and there exist nonzero integers r, s, and t with $\gcd(r, s) = 1$ such that

$$a = (r^2 - s^2)t, \; 2a - 2m = 2rst, \; z = (r^2 + s^2)t \tag{4.16}$$

or

$$a = 2rst, \; 2a - 2m = (r^2 - s^2)t, \; z = (r^2 + s^2)t. \tag{4.17}$$

Clearly, as $z > 0$, we have $t > 0$. Replacing (r, s) by $(-r, -s)$, if necessary, we may suppose that $r > 0$.

We suppose first that (4.16) holds. Then

$$(r^2 - rs - s^2)t = (r^2 - s^2)t - rst = a - (a - m) = m.$$

Now $m = \pm 1$, so $t = 1$ and $r^2 - rs - s^2 = m$. Appealing to the theorem of Wasteels and May, we have

$$r = F_{k+1}, \; s = F_k, \quad m = (-1)^k, \quad \text{if } s > 0,$$
$$r = F_k, \quad s = -F_{k+1}, \; m = (-1)^{k+1}, \; \text{if } s < 0,$$

for some positive integer k. Then, from (4.16), we obtain

$$a = r^2 - s^2 = \begin{cases} F_{k+1}^2 - F_k^2 = F_{k-1}F_{k+2}, & \text{if } s > 0, \\ F_k^2 - F_{k+1}^2 = -F_{k-1}F_{k+2}, & \text{if } s < 0. \end{cases}$$

As $a > 0$, we must have $s > 0$ so that $r = F_{k+1}$, $s = F_k$, $m = (-1)^k$ and

$$a = F_{k-1}F_{k+2}. \tag{4.18}$$

Further, from (4.16), we have

$$z = r^2 + s^2 = F_k^2 + F_{k+1}^2 = F_kF_{k+2} + F_{k-1}F_{k+1} \tag{4.19}$$

and

$$a - m = rs = F_kF_{k+1}. \tag{4.20}$$

Also, as $a>0$, we have $F_{k-1} \neq 0$ so $k \neq 1$ and thus $k \geq 2$. From (4.14), we have

$$\begin{aligned} b &= \frac{a}{2}(a+2(a-m)+\varepsilon z) \\ &= \begin{cases} (1/2)F_{k-1}F_{k+2}(F_{k-1}F_{k+2}+2F_kF_{k+1}+F_kF_{k+2}+F_{k-1}F_{k+1}), & \text{if } \varepsilon = 1, \\ (1/2)F_{k-1}F_{k+2}(F_{k-1}F_{k+2}+2F_kF_{k+1}-F_kF_{k+2}-F_{k-1}F_{k+1}), & \text{if } \varepsilon = -1, \end{cases} \\ &= \begin{cases} F_{k-1}F_{k+1}F_{k+2}^2, & \text{if } \varepsilon = 1, \\ F_{k-1}^2F_kF_{k+2}, & \text{if } \varepsilon = -1. \end{cases} \end{aligned}$$

Thus,

$$a^2-2b = \begin{cases} F_{k-1}^2F_{k+2}^2 - 2F_{k-1}F_{k+1}F_{k+2}^2 = -F_{k-1}F_{k+2}^3, & \text{if } \varepsilon = 1, \\ F_{k-1}^2F_{k+2}^2 - 2F_{k-1}^2F_kF_{k+2} = F_{k-1}^3F_{k+2}, & \text{if } \varepsilon = -1, \end{cases}$$

and

$$a^2-b = \begin{cases} F_{k-1}^2F_{k+2}^2 - F_{k-1}F_{k+1}F_{k+2}^2 = -F_{k-1}F_kF_{k+2}^2, & \text{if } \varepsilon = 1, \\ F_{k-1}^2F_{k+2}^2 - F_{k-1}^2F_kF_{k+2} = F_{k-1}^2F_{k+1}F_{k+2}, & \text{if } \varepsilon = -1. \end{cases}$$

Then, from (4.10), (4.11), and (4.12), we obtain

$$c = -F_kF_{k+1}, \quad \text{if } \varepsilon = \pm 1,$$

$$d = \begin{cases} -F_{k-1}F_{k+1}^2F_{k+2}, & \text{if } \varepsilon = 1, \\ F_{k-1}F_k^2F_{k+2}, & \text{if } \varepsilon = -1, \end{cases}$$

$$e = \begin{cases} F_{k-1}F_{k+1}^3F_{k+2}^2, & \text{if } \varepsilon = 1, \\ -F_{k-1}^2F_k^3F_{k+2}, & \text{if } \varepsilon = -1. \end{cases}$$

From (4.13), we get

$$n = \begin{cases} F_{k-1}^2F_{k+1}^4F_{k+2}^4, & \text{if } \varepsilon = 1, \\ -F_{k-1}^4F_k^4F_{k+2}^2, & \text{if } \varepsilon = -1. \end{cases}$$

Then (4.1) gives the factorizations

$$\begin{aligned} x^5+(-1)^kx^4+F_{k-1}^2F_{k+1}^4F_{k+2}^4 &= (x^2+F_{k-1}F_{k+2}x+F_{k-1}F_{k+1}F_{k+2}^2) \\ &\quad \times (x^3-F_kF_{k+1}x^2-F_{k-1}F_{k+1}^2F_{k+2}x+F_{k-1}F_{k+1}^3F_{k+2}^2), \\ x^5+(-1)^kx^4-F_{k-1}^4F_k^4F_{k+2}^2 &= (x^2+F_{k-1}F_{k+2}x+F_{k-1}^2F_kF_{k+2}) \\ &\quad \times (x^3-F_kF_{k+1}x^2+F_{k-1}F_k^2F_{k+2}x-F_{k-1}^2F_k^3F_{k+2}), \end{aligned}$$

and two more obtained by changing x to $-x$. These are the factorizations given in the statement of Theorem 3.

We now suppose that (4.17) holds. Then

$$(r^2-4rs-s^2)t = (r^2-s^2)t - 4rst = (2a-2m) - 2a = -2m.$$

As $m = \pm 1$ and $t>0$, we have $t=1$ or $t=2$. If $t=1$, then

$$r^2-4rs-s^2 = -2m. \tag{4.21}$$

Hence, $r \equiv r^2 \equiv s^2 \equiv s \pmod 2$. But $\gcd(r, s) = 1$, so $r \equiv s \equiv 1 \pmod 2$. Then $r^2 - s^2 \equiv 0$ (mod 4), which contradicts (4.21). Therefore, we must have $t = 2$, in which case $r^2 - 4rs - s^2 = -m$, so that

$$(2r)^2 - (2r)(r+s) - (r+s)^2 = -m.$$

As $a > 0$, $r > 0$, and $t > 0$, we see from (4.17) that $s > 0$. Thus, $2r$ and $r + s$ are positive integers, and so, by the theorem of Wasteels and May, we have

$$2r = F_{k+1}, \quad r + s = F_k, \quad -m = (-1)^k,$$

for some positive integer k. Thus,

$$r = \tfrac{1}{2}F_{k+1}, \quad s = \tfrac{1}{2}F_{k-2}, \quad m = (-1)^{k+1}.$$

As $s \neq 0$, we see that $k \neq 2$. Now $2|F_h \Leftrightarrow 3|h$ (see [6], p. 32), so as r and s are integers, we have

$$r = \tfrac{1}{2}F_{3l+3}, \quad s = \tfrac{1}{2}F_{3l}, \quad m = (-1)^{l+1},$$

for some integer $l \geq 1$. Hence, by (4.17), we have

$$a = F_{3l}F_{3l+3}, \tag{4.22}$$

$$z = \tfrac{1}{2}(F_{3l}^2 + F_{3l+3}^2) = F_{3l+1}F_{3l+3} + F_{3l}F_{3l+2}, \tag{4.23}$$

and

$$a - m = \tfrac{1}{4}(F_{3l+3}^2 - F_{3l}^2) = F_{3l+1}F_{3l+2}. \tag{4.24}$$

Comparing (4.22), (4.23), and (4.24) to (4.18), (4.19), and (4.20), respectively, we see that the possibility (4.17) just leads to a special case $k = 3l + 1$ ($l \geq 1$) of the previous case and, therefore, does not lead to any new factorizations.

The discriminant of $x^2 + \theta F_{k-1}F_{k+2}x + F_{k-1}F_{k+1}F_{k+2}^2$ is

$$\begin{aligned} F_{k-1}^2F_{k+2}^2 - 4F_{k-1}F_{k+1}F_{k+2}^2 &= F_{k-1}F_{k+2}^2(F_{k-1} - 4F_{k+1}) \\ &= -F_{k-1}F_{k+2}^2(3F_{k-1} + 4F_k), \end{aligned}$$

which is negative for $k \geq 2$. Hence, $x^2 + \theta F_{k-1}F_{k+2}x + F_{k-1}F_{k+1}F_{k+2}^2$ is irreducible. Similarly, the discriminant of $x^2 + \theta F_{k-1}F_{k+2}x + F_{k-1}^2F_kF_{k+2}$ is

$$\begin{aligned} F_{k-1}^2F_{k+2}^2 - 4F_{k-1}^2F_kF_{k+2} &= F_{k-1}^2F_{k+2}(F_{k+2} - 4F_k) \\ &= F_{k-1}^2F_{k+2}(F_{k+1} - 3F_k) \\ &= F_{k-1}^2F_{k+2}(F_{k-1} - 2F_k) \\ &= -F_{k-1}^2F_{k+2}(F_{k-1} + 2F_{k-2}), \end{aligned}$$

which is negative for $k \geq 2$. Thus, $x^2 + \theta F_{k-1}F_{k+2}x + F_{k-1}^2F_kF_{k+2}$ is irreducible. To complete the proof of Theorem 3, it remains to show that the cubic polynomials

$$x^3 - \theta F_kF_{k+1}x^2 - F_{k-1}F_{k+1}^2F_{k+2}x + \theta F_{k-1}F_{k+1}^3F_{k+2}^2$$

and

$$x^3 - \theta F_kF_{k+1}x^2 + F_{k-1}F_k^2F_{k+2}x - \theta F_{k-1}^2F_k^3F_{k+2}$$

are irreducible over the rational field Q for $k \geq 2$ and $\theta = \pm 1$. This is done in the next section. It clearly suffices to treat only the case $\theta = 1$.

5. IRREDUCIBILITY OF TWO CUBIC POLYNOMIALS

In this section we prove that the two cubic polynomials

$$f(x) = x^3 - F_k F_{k+1} x^2 - F_{k-1} F_{k+1}^2 F_{k+2} x + F_{k-1} F_{k+1}^3 F_{k+2}^2 \tag{5.1}$$

and

$$g(x) = x^3 - F_k F_{k+1} x^2 + F_{k-1} F_k^2 F_{k+2} x - F_{k-1}^2 F_k^3 F_{k+2} \tag{5.2}$$

are irreducible over the rationals for $k \geq 2$. Before proving this (see Theorem 4 below), we prove three lemmas.

Lemma 1: If N is a nonzero integer, then the quintic equation $x^5 + x^4 + N = 0$ has exactly one real root.

Proof: The function $F(x) = x^5 + x^4 + N$ has a local maximum at $x = -4/5$ and a local minimum at $x = 0$. There are no other local maxima or local minima. Clearly, $F(-4/5) = N + 4^4/5^5$ and $F(0) = N$. As N is a nonzero integer, we cannot have $N \leq 0 \leq N + 4^4/5^5$. Hence, either $N > 0$ or $N + 4^4/5^5 < 0$. If $N > 0$, the curve $y = F(x)$ meets the x-axis at exactly one point x_0 $(x_0 < -4/5)$. If $N + 4^4/5^5 < 0$, the curve $y = F(x)$ meets the x-axis at exactly one point x_1 $(x_1 > 0)$. Hence, $F(x) = 0$ has exactly one real root.

Lemma 2: For $k \geq 2$, each of the quintic polynomials

$$A(x) = x^5 + (-1)^k x^4 + F_{k-1}^2 F_{k+1}^4 F_{k+2}^4 \tag{5.3}$$

and

$$B(x) = x^5 + (-1)^k x^4 - F_{k-1}^4 F_k^4 F_{k+2}^2 \tag{5.4}$$

has exactly one real root.

Proof: As $k \geq 2$, $(-1)^k F_{k-1}^2 F_{k+1}^4 F_{k+2}^4$ is a nonzero integer. Hence, by Lemma 1, the quintic polynomial $Q(y) = y^5 + y^4 + (-1)^k F_{k-1}^2 F_{k+1}^4 F_{k+2}^4$ has exactly one real root. Thus, the quintic polynomial $A(x) = (-1)^k Q((-1)^k x)$ has exactly one real root. The quintic polynomial $B(x)$ can be treated similarly.

Lemma 3: For $k \geq 2$, each of the cubic polynomials $f(x)$ and $g(x)$ has exactly one real root.

Proof: From (1.1), (1.2), (5.1), (5.2), (5.3), and (5.4), we have

$$A(x) = (x^2 + F_{k-1} F_{k+2} x + F_{k-1} F_{k+1} F_{k+2}^2) f(x) \quad \text{and} \quad B(x) = (x^2 + F_{k-1} F_{k+2} x + F_{k-1}^2 F_k F_{k+2}) g(x).$$

Since the two quadratics have no real roots, the result follows from Lemma 2.

Theorem 4: For $k \geq 2$ the cubic polynomials $f(x)$ and $g(x)$ are irreducible over the rationals.

Proof: Suppose $f(x)$ is reducible over the rationals. Then, by Lemma 3, $f(x)$ has exactly one real root, which must be rational and, in fact, an integer. Thus,

$$f_1(x) = \frac{1}{F_{k+1}^3} f(F_{k+1} x) = x^3 - F_k x^2 - F_{k-1} F_{k+2} x + F_{k-1} F_{k+2}^2$$

has exactly one real root, which must be an integer. Hence,

$$f_2(x) = f_1(x - F_{k+1}) = x^3 - (3F_{k+1} + F_k)x^2 + F_{2k+3}x + (-1)^k F_{k+2}$$

has exactly one real root r, which must be an integer. If k is even, then $f_2(0) = F_{k+2} > 0$ and

$$f_2(-1) = -1 - 3F_{k+1} - F_k - F_{2k+3} + F_{k+2} < F_{k+2} - F_{2k+3} < 0,$$

so $-1 < r < 0$, which is impossible. If k is odd, then $f_2(0) = -F_{k+2} < 0$ and

$$\begin{aligned} f_2(1) &= 1 - 3F_{k+1} - F_k + F_{2k+3} - F_{k+2} \\ &= 1 + (F_{2k+1} - F_{k+3}) + (F_{2k+2} - F_{k+3}) \geq 1 > 0, \end{aligned}$$

so $0 < r < 1$, which is impossible. Hence, $f(x)$ is irreducible over Q.

We now turn to $g(x)$. Suppose $g(x)$ is reducible over Q. Then, by Lemma 3, $g(x)$ has exactly one real root, which must be rational and, in fact, integral. Thus,

$$g_1(x) = \frac{1}{F_k^3} g(F_k x) = x^3 - F_{k+1}x^2 + F_{k-1}F_{k+2}x - F_{k-1}^2 F_{k+2}$$

has exactly one real root, which must be an integer. Therefore,

$$g_2(x) = g_1(x + F_k) = x^3 + (F_k + F_{k-2})x^2 + F_{2k-1}x + (-1)^{k-1}F_{k-1}$$

has exactly one real root s, which must be an integer. If k is even, then $g_2(0) = -F_{k-1} < 0$ and

$$\begin{aligned} g_2(1) &= 1 + F_k + F_{k-2} + F_{2k-1} - F_{k-1} \\ &\geq 1 + F_{k-2} + F_{2k-1} > 0, \end{aligned}$$

so that $0 < s < 1$, which is impossible. If k is odd, then $g_2(0) = F_{k-1} > 0$ and

$$\begin{aligned} g_2(-1) &= -1 + F_k + F_{k-2} - F_{2k-1} + F_{k-1} \\ &= -1 + 2F_k - 2F_{2k-3} - F_{2k-4} \\ &\leq -1 - 2(F_{2k-3} - F_k) \leq -1 < 0, \end{aligned}$$

so that $-1 < s < 0$, which is impossible. Hence, $g(x)$ is irreducible over Q.

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