SOLVING GENERALIZED FIBONACCI RECURRENCES

D. A. Wolfram
Dept. of Computer Science, The Australian National University,
CSIT Building, North Road, Canberra, ACT 0200 Australia
(Submitted June 1996-Final Revision July 1997)

1. GENERALIZATIONS

We consider finding a Binet formula for the continuous function $f : \Re \to \Im$ which has the property

$$f(x) = \sum_{1 \le l \le k} f(x-l), \tag{1}$$

where

- k is a given integer and $k > 1$,
- either $f(0), \ldots, f(k-1)$ are given initial values,
- or $f : [0, k) \to \Im$ is a given continuous *initial function* where

$$\lim_{x \to k} f(x) = \sum_{0 \le x \le k-1} f(x).$$

This generalizes the Fibonacci sequence in new ways. Instead of viewing the sequence as an automorphism on the integers, its domain becomes the reals. The Binet formula also allows the initial values to be arbitrary values, possibly complex ones. Instead of having k initial values for the function of order k, we also allow an *initial function* which is defined on the interval $[0, k)$.

When only k initial values are given, there can be many possible functions f. However, the following can be shown by induction.

Lemma 1.1: Given an initial function, f is uniquely defined on $\Re$, and if

$$\lim_{x \to k} f(x) = \sum_{0 \le x \le k-1} f(x),$$

then f is continuous.

2. RELATED WORK

In 1961, Horadam wrote that generalizations of Fibonacci's sequence either involved changes to the Fibonacci recurrence or allowed its initial values to be changed or, possibly, a combination of these [10].

Since then, the main contributions to a general theory seem to involve generalizations of the Fibonacci recurrence [17], [20]:

$$f(x) = \sum_{1 \le l \le k} f(x-l). \tag{2}$$

When $k = 3$ and $f(0), f(1)$, and $f(2)$ are arbitrary constants, this is the recurrence of the *generalized Tribonacci sequence*. The *Tetranacci* or *Quadranacci sequence* is similarly defined when $k = 4$.

Direct evaluation of equation (2) can have exponential complexity. Burstall and Darlington [2] gave a linear algorithm for computing Fibonacci numbers as an example of their program transformation methods:

$$\begin{aligned}
&\mathrm{f}(0) \Leftarrow 1\\
&\mathrm{f}(1) \Leftarrow 1\\
&\mathrm{f}(\mathrm{x}+2) \Leftarrow \mathrm{u}+\mathrm{v}, \text{where } \langle \mathrm{u}, \mathrm{v}\rangle = \mathrm{g}(\mathrm{x})\\
&\mathrm{g}(0) \Leftarrow \langle 1,\ 1\rangle\\
&\mathrm{g}(\mathrm{x}+1) \Leftarrow \langle \mathrm{u}+\mathrm{v}, \mathrm{u}\rangle, \text{where } \langle \mathrm{u}, \mathrm{v}\rangle = \mathrm{g}(\mathrm{x})
\end{aligned}$$

This approach can, of course, be generalized by allowing different initial values and letting $k > 2$. For example, if $\mathtt{f(0)}$ is defined to be 0 instead, we have that $\mathtt{f(n)}$ is the n^{th} Fibonacci number. Given an efficient implementation of exponentiation, using the Binet formula

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

for the same task can have lower complexity.

A similar formula, where $k = 2$ and the initial values are arbitrary, was given by Horadam [10]. A Binet formula for the recursive sequence of order k was given by Miles [13] for the special case $f(x) = 0$, where $0 \le x \le k-2$ and $f(k-1) = 1$. Spickerman and Joyner [18] gave another solution for the special case $f(0) = 1$, and $f(x) = 2^{x-1}$ for $1 \le x \le k-1$. Our approach subsumes these results as special cases.

We have also derived a solution to equation (1), where an arbitrary initial function is specified. This does not seem to have been considered before.

In the next section, we discuss properties of the characteristic equation, the coefficients of generalized Binet formulas, and solutions that use the initial values, and the initial function. When the initial values are given, we present two methods of solution: one uses Binet formulas and the other uses an exponential generating function and the Laplace Transform. We use the latter method to find solutions when $2 \le k \le 4$. They are equivalent to those found with Binet formulas, but they are more complicated and do not involve complex roots.

3. THE CHARACTERISTIC EQUATION

We consider properties of the characteristic equation associated with Fibonacci recurrences of equation (1). These properties include its discriminant, location of roots, reducibility, and solvability in radicals. Several of the results here are used in later sections.

Equation (1) is a homogeneous linear difference equation. Its characteristic equation* is given below:

$$y^k - \sum_{0 \le l < k} y^l = 0. \tag{3}$$

The form of the general solution of such difference equations depends on whether the roots of its characteristic equation are simple [12]. We define the *characteristic function* of order k to be $c_k(y) = y^k - \sum_{0 \le l < k} y^l$.

* See Liu [12], §3-2, for example.

Lemma 3.1: For every solution r of c_k,

$$c_j(r)+\frac{c_{k-j}(r)}{r^{k-j}}=1,$$

where $0\le j\le k$.

Proof: This follows from the definition of c_k. □

Corollary 3.2: For every solution r of c_k,

$$c_j(r)=\sum_{1\le i\le k-j} r^{-i}.$$

Theorem 3.3: The discriminant of the characteristic equation is

$$(-1)^{\frac{k(k+1)}{2}}\left[\frac{(k+1)^{k+1}-2(2k)^k}{(k-1)^2}\right]$$

when $k>1$.

Proof: Let the resultant of $c_k(y)$ and $\frac{dc_k}{dy}$ be $R(c,c')$. The discriminant of the characteristic equation is $(-1)^{\frac{k(k-1)}{2}}R(c,c')$ [11]. The resultant when $k=3$ is

$$R(c,c')=\begin{vmatrix}1&-1&-1&-1&0\\0&1&-1&-1&-1\\3&-2&-1&0&0\\0&3&-2&-1&0\\0&0&3&-2&-1\end{vmatrix}.$$

This can be simplified by partial Gaussian elimination. First, we interchange elements by moving element $a_{i,j}$ to element $a_{2k-i,2k-j}$, where $1\le i,j\le 2k-1$. This does not change the sign of the determinant. In the example above, we obtain

$$R(c,c')=\begin{vmatrix}-1&-2&3&0&0\\0&-1&-2&3&0\\0&0&-1&-2&3\\-1&-1&-1&1&0\\0&-1&-1&-1&1\end{vmatrix}.$$

Subtracting row 1 from row 4 and row 2 from row 5 yields

$$R(c,c')=\begin{vmatrix}-1&-2&3&0&0\\0&-1&-2&3&0\\0&0&-1&-2&3\\0&1&-4&1&0\\0&0&-1&-4&1\end{vmatrix}.$$

If we then add row 2 to row 4 and row 3 to row 5, we obtain

$$R(c,c')=\begin{vmatrix}-1&-2&3&0&0\\0&-1&-2&3&0\\0&0&-1&-2&3\\0&0&-6&4&0\\0&0&0&-6&4\end{vmatrix}.$$

In general, the last two row operations above can be defined as the replacement of element $a_{k+i,j}$ by $a_{k+i,j} - a_{i,j} + a_{i+1,j}$, where $1 \le i \le k-1$ and $1 \le j \le 2k-1$. Rows $k+1$ to $2k-1$ in these determinants have the form

$$0 \;\cdots\; 0 \;\; -2k \;\; k+1 \;\; 0 \;\cdots\; 0,$$

where row $l: k+1 \le l \le 2k-1$ has 0 in columns 1 to $l-2$. By induction on k, we can show that, for all $k \ge 1$,

$$R(c, c') = (-1)^k \left[-\frac{(2k)^k}{2} + \sum_{0 \le i \le k-2} (i+1)(2k)^i (k+1)^{k-1-i} \right].$$

The following identity can be used to simplify the summation in this expression:

$$\sum_{0 \le l \le n-1} (a + ld)x^l = \frac{a - (a + (n-1)d)x^n}{1-x} + \frac{dx(1 - x^{n-1})}{(1-x)^2}.$$

With $x = \frac{2k}{k+1}$ and $n = k-1$, we obtain

$$R(c, c') = (-1)^k \left[\frac{(k+1)^{k+1} - 2(2k)^k}{(k-1)^2} \right]$$

when $k > 1$, and the result follows. □

Miles [13] and Miller [14] have shown that the characteristic equation has simple roots. Corollary 3.4 below shows this by different means. Its proof will be used in the proof of Theorem 3.9 below.

Corollary 3.4: The characteristic equation has simple roots.

Proof: It suffices to show that $R(c, c') \ne 0$. The resultant equals -5 when $k = 2$, and 44 when $k = 3$. It could only be zero if $(k+1)^{k+1} = 2(2k)^k$, which occurs if $(k+1)\log_2(k+1) - 1 - k - k\log_2 k = 0$. Now $\log_2(k+1) - \log_2 k < 0.6$ when $k > 2$, so that

$$(k+1)\log_2(k+1) - 1 - k - k\log_2 k < \log_2 k - 0.4(k+1).$$

When $1 \le k \le 4$, $\log_2 k - 0.4(k+1) \le 0$. The derivative of $\log_2 k - 0.4(k+1)$ is negative when $k \ge 4$. Thus, $(k+1)\log_2(k+1) - 1 - k - k\log_2 k < 0$ and $R(c, c') \ne 0$ when $k > 1$, as required. □

We call the k roots of equation (3), $r_1, \ldots, r_k$.

Corollary 3.5 The general solution to equation (1) when x is an integer has the form

$$f(x) = \sum_{1 \le i \le k} C_i r_i^x, \tag{4}$$

where the C_i are constant coefficients.

To find a solution to equation (1), we need to find a version of this summation where x can be a real number.

The following lemma identifies the locations and limits of the roots of the characteristic equation more precisely than previous results by Miles [13] and Miller [14].

Lemma 3.6: The characteristic equation $y^k - \sum_{0\le l<k} y^l = 0$ has one positive real root in the interval (1, 2). This root approaches 2 as k approaches infinity, and it is greater than $2(1-2^{-k})$.

It has one negative real root in the interval (−1, 0) when k is even. This root and each complex root r has modulus $3^{-k} < |r| < 1$.

Proof: By Descartes' Rule of Signs [6], the characteristic equation has one variation, and so has at most one positive real root. There must be just one positive root because the characteristic function is $-k+1$ when $y=1$, and 1 when $y=2$. Another proof of this follows immediately from Pólya and Szegö ([15], Vol. I, Pt. III, Prob. 16). The characteristic function c_{k+1} is

$$y\left(y^k - \frac{y^k-1}{y-1}\right) - 1.$$

At $y=r$, this equals −1. By Lemma 3.6, there is only one positive root, so that positive root of c_{k+1} is greater than r. Hence, r is always greater than $\frac{1}{2}$ when $k \ge 2$, so that

$$r^k - \frac{\frac{1}{2}^k - 1}{-\frac{1}{2}}$$

is always positive. The root r must be less than 2 because the characteristic equation always equals 1 when $x = 2$. Therefore, r lies between $2(1-2^{-k})$ and 2. As k approaches infinity, r approaches 2.

If we replace y by $-x$, the characteristic function is $-x^k - \frac{x^k+1}{x+1}$ when k is odd. This is negative when $x \ge 0$, and so the characteristic equation does not have any negative real roots.

When k is even, replacing y by $-x$ in the characteristic function gives $x^k + \frac{x^k-1}{x+1}$. This is positive when $x \ge 1$, so there is at least one negative root of the characteristic equation between −1 and 0. The derivative of the preceding function is

$$\frac{kx^{k+1} + (3k-1)x^k + 2kx^{k-1} + 1}{(x+1)^2},$$

which is positive when $x \ge 0$. Therefore, $x^k + \frac{x^k-1}{x+1}$ is strictly increasing when $x \ge 0$. It follows that there is only one negative root of the characteristic equation.

We now consider the complex roots of the characteristic equation. From Miles [13] and Miller [14], each of them has modulus less than one. For every root r, $|r| = |r-2|^{-1/k}$. In the region where $|z|<1$, we have $1<|z-2|<3$, and so each of the complex roots and the negative real root satisfies $3^{-1/k} < |r| < 1$. □

Corollary 3.7: $\lim_{x\to\infty} \frac{f(x+1)}{f(x)} = r_1$ and $\lim_{x\to\infty} \frac{f(x)}{r_1^x} = C_1$.

Proof: This follows immediately from Corollary 3.5, and that $|r_i| < 1$ for $i: 2 \le i \le k$. □

Corollary 3.8: c_k is irreducible over the rationals where $k > 1$.

Proof: By Gauss's lemma, the irreducibility of c_k over the rationals is equivalent to its irreducibility over the integers [11]. If c_k were reducible, the roots of one of its factors would all have moduli that are strictly less than 1. The product of these roots cannot be an integer. This leads to a contradiction because the modulus of the product of these roots must equal the modulus of the constant term of this factor.* □

We now give a series that can be used to evaluate the positive real root of the characteristic equation.

Theorem 3.9: Let $2(1-\varepsilon_k)$ be the positive root of the characteristic equation. Then

$$\varepsilon_k = \sum_{i\geq 1} \binom{(k+1)i-2}{i-1} \frac{1}{i2^{(k+1)i}}.$$

Proof: We have

$$\varepsilon_k = \sum_{i\geq 0} \binom{(k+1)i+(k-1)}{i} \frac{1}{(i+1)2^{(k+1)(i+1)}}.$$

Define

$$\varepsilon_k(z) = \sum_{i\geq 0} \binom{(k+1)i+(k-1)}{i} \frac{z^{i+1}}{i+1}.$$

This is equivalent to the previous expression when $z = \left(\frac{1}{2}\right)^{k+1}$. We have

$$\frac{d\varepsilon_k(z)}{dz} = \sum_{i\geq 0} \binom{(k+1)i+(k-1)}{i} z^i.$$

Identity 29 on page 713 of Prudnikov, Brychkov, and Marichev [16] states**

$$\sum_{k=0}^{\infty} \frac{\Gamma(k\nu+\mu)}{k!\,\Gamma(k\nu-k+\mu)} x^k = \frac{y^\mu}{(1-\nu)y+\nu}, \tag{5}$$

where

$$x = \frac{y-1}{y^\nu} \quad \text{and} \quad |x| = \left|\frac{(\nu-1)^{\nu-1}}{\nu^\nu}\right|.$$

If we rename k by i, and then let $\nu = k+1$, $\mu = k$, $x = z$, and $y = x$, we have

$$\frac{d\varepsilon_k(z)}{dz} = \frac{x^k}{k+1-kx},$$

where

$$z = \frac{x-1}{x^{k+1}}, \text{ provided that } |z| < \left|\frac{k^k}{(k+1)^{k+1}}\right|. \tag{6}$$

When $z = \left(\frac{1}{2}\right)^{k+1}$, this simplifies to $2(2k)^k - (k+1)^{k+1} > 0$. It is remarkable that this is the same condition as in the proof of Corollary 3.4, so that it always holds when $k > 1$.

---

* David Boyd told me of this proof. It is known from the theory of Pisot numbers [1].

** Prudnikov, Brychkov, and Marichev [16] seem to refer to Gould [9] for this result. Gould gives a more restricted form where combinations rather than Gamma functions are used (Identity 1.120 on p. 15). Gould, in turn, apparently refers to the 1925 German edition of Pólya and Szegö [15]. The identity appears as a solution to problem **216** of Part III of Volume I of the 1972 English translation of that work [15]. The convergence condition (6) is discussed by Gould [8].

Now

$$\frac{dz}{dx} = \frac{k+1-kx}{x^{k+2}}$$

so that

$$\frac{d\varepsilon_k(z)}{dz}\frac{dz}{dx} = \frac{1}{x^2}.$$

We have

$$\int_0^{(\frac{1}{2})^{k+1}} \frac{d\varepsilon_k(z)}{dz}dz = \int_1^{x_0} \frac{1}{x^2}dx,$$

where

$$\frac{x_0-1}{x_0^{k+1}} = \left(\tfrac{1}{2}\right)^{k+1}.$$

Since $\frac{dz}{dx} = \frac{k+1-kx}{x^{k+2}}$, the value of z as a function of x is increasing when $1 \le x \le \frac{k+1}{k}$. In this interval, z increases from 0 to $\frac{k^k}{(k+1)^{k+1}}$. We have shown that condition (6) holds when $z = \left(\frac{1}{2}\right)^{k+1}$. This implies that $< x_0 < \frac{k+1}{k}$, and for all $x: 1 \le x \le x_0$, condition (6) is satisfied.

Therefore, $\varepsilon_k = 1 - \frac{1}{x_0}$ and the positive root of the characteristic equation is $\frac{2}{x_0}$. To check this, we can write the characteristic equation as

$$\frac{y^k(y-2)+1}{y-1} = 0,$$

which holds when $y^k(y-2)+1=0$ and $y \ne 1$. On substitution of $y = \frac{2}{x_0}$, we obtain

$$\left(\frac{2}{x_0}\right)^{k+1} - 2\left(\frac{2}{x_0}\right)^k + 1 = 0.$$

This is equivalent to $\frac{x_0-1}{x_0^{k+1}} = \left(\frac{1}{2}\right)^{k+1}$, as required. □

Remark 3.10: Condition (6) and the one for equation (5) [16] should be strengthened. We have used a value $x_0: 1 < x_0 < \frac{k+1}{k}$ such that $\frac{x_0-1}{x_0^{k+1}} = \left(\frac{1}{2}\right)^{k+1}$, but we could have chosen $x_0 = 2$ instead. This value also satisfies the condition, but in general,

$$\frac{x_0^k}{k+1-kx_0} \ne \frac{2^k}{1-k}.$$

3.1 Solvability in Radicals

We now consider the roots of specific characteristic equations. When $k=2$, we have $r_1 = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{5}}{2}$, or $-r_1^{-1}$. Approximate values of r_1 and r_2 are 1.618033988749895 and −0.618033988749895, respectively.

When $k = 3$, we let the real root of equation (3) be

$$r_1 = \frac{1+(19-3\sqrt{33})^{1/3}+(19+3\sqrt{33})^{1/3}}{3}.$$

This was found by using "Cardan's Method" of 1545, due to Ferro and Tartaglia, for the solution of the general cubic equation (see [6], [7]). This constant r_1 will be used to find a solution to equation (1) above. The complex solutions are $-\omega p-\omega^2 q+\frac{1}{3}$ and $-\omega^2 p-\omega q+\frac{1}{3}$, where

$$\omega=\frac{-1+\sqrt{3}i}{2},\ \omega^2=\frac{-1-\sqrt{3}i}{2},\ p=\frac{-(19-3\sqrt{33})^{1/3}}{3},\ q=\frac{-(19+3\sqrt{33})^{1/3}}{3}.$$

An approximate value of r_1 is 1.83928675521416. The approximate values of the complex roots are $-0.419643377607081\pm 0.606290729207199i$.

Similarly, when $k=4$, the two real solutions of (3) are given by

$$\left(p_1+\frac{1}{4}\right)\pm\sqrt{\left(p_1+\frac{1}{4}\right)^2-\frac{2\lambda_1}{p_1}\left(p_1+\frac{1}{4}\right)+\frac{7}{24p_1}+\frac{1}{6}},$$

where

$$\lambda_1=\frac{(3\sqrt{1689}-65)^{1/3}-(3\sqrt{1689}+65)^{1/3}}{12\cdot 2^{1/3}}\quad\text{and}\quad p_1=\sqrt{\lambda_1+\frac{11}{48}}.$$

The complex roots are given by

$$\left(\frac{1}{4}-p_1\right)\pm\sqrt{\left(\frac{1}{4}-p_1\right)^2+\frac{2\lambda_1}{p_1}\left(\frac{1}{4}-p_1\right)-\frac{7}{24p_1}+\frac{1}{6}}.$$

These were found by Ferrari's Solution to the general quartic (see [6], [7]). Approximations of the real solutions that we call r_1 and r_2 are 1.92756197548293 and -0.77480411321543, respectively. The approximate values of the complex solutions are $-0.0763789311337454\pm 0.814703647170387i$.

Lemma 3.11: There are no solutions in radicals to the characteristic equation when $5\le k\le 11$.

Proof: The Galois group of the characteristic equation is S_k when $1\le k\le 11$. These groups were found by using Magma* [3], and they are not soluble [11]. □

We conjecture that the Galois group of the characteristic equation is also S_k when $k>11$. In general, computing the Galois group of a polynomial currently seems to be intractable when $k\ge 12$ (see [19]).

4. THE COEFFICIENTS

We consider the problem of finding the coefficients C_i in the equation

$$f(x)=\sum_{1\le i\le k} C_i r_i^x, \tag{7}$$

where $f(0),\ldots,f(k-1)$ are given. This is the problem of finding a general solution of a homogeneous linear difference equation whose characteristic equation has simple roots.

* These computations were done by John Cannon. Robert Low also told me independently that Maple [4] gave the same answers where $5\le k\le 8$. Values of k outside this range were not used.

We use the elementary symmetric polynomials ([7], [11]) σ_i^{k-1} defined over $\{y_1, \ldots, y_{k-1}\}$, where $1 \le i \le k-1$, and define $\sigma_0^{k-1} = 1$. The coefficient C_i is then given by the function

$$h(y_1, \ldots, y_k) = \frac{\sum_{0\le j<k} (-1)^j f(k-1-j)\sigma_j^{k-1}}{\prod_{1\le j<k} (y_k - y_j)}, \tag{8}$$

where $y_k = r_i$ and $y_1, \ldots, y_{k-1}$ are assigned, respectively, to the other $k-1$ roots of equation (3) in any order. Equation (8) can be verified by induction on k. The formula was derived by Gaussian elimination and back substitution on systems such as

$$\begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} f(0) \\ f(1) \\ f(2) \end{bmatrix}$$

when $k = 3$. More generally, the leftmost $k \times k$ matrix has elements $a_{i,j} = r_j^{i-1}$. The determinant of this matrix is the Vandermonde determinant. Lang [11] in Exercise 33(c) of Chapter V discusses how this determinant can be used to find the coefficients, but no explicit formula is given.

Example 4.1: When $k = 4$, we have

$$f(x) = C_1r_1^x + C_2r_2^x + C_3r_3^x + C_4r_4^x,$$

where the function $h(y_1, y_2, y_3, y_4)$ is

$$\frac{f(3) - (y_1 + y_2 + y_3)f(2) + (y_1y_2 + y_2y_3 + y_3y_1)f(1) - y_1y_2y_3f(0)}{(y_4 - y_1)(y_4 - y_2)(y_4 - y_3)}.$$

Thus, $f(x)$ is

$$h(r_2, r_3, r_4, r_1)r_1^x + h(r_1, r_3, r_4, r_2)r_2^x + h(r_1, r_2, r_4, r_3)r_3^x + h(r_1, r_2, r_3, r_4)r_4^x.$$

Some special cases have appeared in the literature. Horadam [10] presented a Binet formula called equation (δ), which is equivalent to the one below, where $k = 2$, $f(0) = q$, and $f(1) = p$:

$$f(x) = \frac{1}{2\sqrt{5}}(2(p - qr_2)r_1^x - 2(p - qr_1)r_2^x).$$

From equation (8), we find

$$h(y_1, y_2) = \frac{f(1) - y_1f(0)}{y_2 - y_1},$$

so that

$$C_1 = \frac{p - r_2q}{\sqrt{5}} \quad \text{and} \quad C_2 = \frac{p - r_1q}{-\sqrt{5}}$$

in agreement with Horadam's result.

Miles [13] discussed the special case in which $f(x) = 0$, where $0 \le x \le k-2$ and $f(k-1) = 1$. Equation (7) becomes

$$h(y_1, \ldots, y_k) = \frac{1}{\prod_{1 \le j < k} (y_k - y_j)}$$

in agreement with his equation (2″).

Spickerman and Joyner [18] considered the special case in which $f(0) = 1$ and $f(x) = 2^{x-1}$, for $1 \le x \le k-1$. Their solution is

$$C_i = \frac{r_i^{k+1} - r_i^k}{2r_i^k - (k+1)}. \tag{9}$$

This again is equivalent to a particular solution using equation (8).

Theorem 4.2: When $f(0) = 1$ and $f(x) = 2^{x-1}$, for $1 \le x \le k-1$, equation (9) is equivalent to equation (8).

Proof: The numerator of equation (8) with Spickerman and Joyner's initial values is equal to $\frac{1}{2}\left(\frac{1}{2-r_i} + \frac{1}{r_i}\right)$. This follows from the observation that $\prod_{1 \le i \le k} r_i = (-1)^{k-1}$, $c_k(2) = 1$, and

$$\frac{c_k(2)}{2 - r_i} = \frac{-1}{r_i} + 2 \sum_{0 \le j < k} (-1)^j f(k-1-j) \sigma_j^{k-1}.$$

The expression for the numerator simplifies to y_k^{k-1} by use of the identity $r^k(2-r) = 1$, where r is any root of the characteristic equation.

The denominator of equation (8) can be expressed in terms of r_i by using the property that, for this problem, $\sigma_j^k = (-1)^{j-1}$, where $1 \le j \le k$ [11]. By induction on k, we can show that

$$\prod_{1 \le j < k} (y_k - y_j) = (k-1) y_k^{k-1} + \frac{1}{y_k} - \sum_{1 \le l \le k-2} l y_k^l.$$

The summation can be removed by use of the identity

$$\sum_{0 \le l \le n-1} (a + ld) x^l = \frac{a - (a + (n-1)d) x^n}{1 - x} + \frac{dx(1 - x^{n-1})}{(1-x)^2}$$

to give

$$\prod_{1 \le j < k} (y_k - y_j) = (k-1) y_k^{k-1} + \frac{1}{y_k} - \frac{(k-2) y_k^{k-1}}{1 - y_k} - \frac{y_k (1 - y_k^{k-2})}{(1 - y_k)^2}.$$

After some algebraic simplifications involving uses of the identity $r^{k+1} - r^k = r^k - 1$, the expression for the denominator can be simplified to

$$\frac{y_k^{k-1}(y_k^{k+1} - k)}{y_k^k - 1}.$$

Hence, in this case, equation (8) is equivalent to

$$\frac{y_k^k - 1}{(y_k^{k+1} - k)}.$$

Equation (9) can be derived from this by using the above identity, as required. □

The formula for the coefficient C_i of a generalized Fibonacci recurrence can be expressed merely in terms of the initial values and the root r_i of the characteristic equation.

Corollary 4.3: In the case of the recurrence in equation (1), equation (8) is equivalent to

$$h(y_1, \ldots, y_k) = \frac{(y_k^k - 1) \sum_{0 \le j < k} f(k-1-j) c_j(y_k)}{y_k^{k-1}(y_k^{k+1} - k)}.$$

Proof: This follows from the proof of Theorem 4.2 and by induction on k to show that

$$\sigma_j^{k-1} = (-1)^j \left(y_k^j - \sum_{0 \le i < j} y_k^i \right),$$

where $0 \le j < k$. □

Since the coefficient C_i only depends on the root r_i and the k initial values, we write $h(y)$ instead of $h(y_1, \ldots, y_k)$. From Corollaries 3.2 and 4.3, we obtain

$$\sigma_j^{k-1} = (-1)^j \sum_{1 \le i \le k-j} y_k^{-i} \tag{10}$$

and

$$h(y) = \frac{(y^k - 1) \sum_{0 \le j < k} f(k-1-j) \sum_{1 \le i \le k-j} y^{-i}}{y^{k-1}(y^{k+1} - k)}. \tag{11}$$

Lemma 4.4: Suppose that the k initial values of f are real numbers. When k is even,

$$f(x) = h(r_1)r_1^x + h(r_2)r_2^x + \sum_{1 \le i \le \frac{k}{2}-1} 2v_i w_i^x \cos(\theta_i + \gamma_i x),$$

where v_i and w_i are real constants, and $-\pi < \theta_i,\ \gamma_i \le \pi$. When k is odd,

$$f(x) = h(r_1)r_1^x + \sum_{1 \le i \le \frac{k-1}{2}} 2v_i w_i^x \cos(\theta_i + \gamma_i x).$$

Proof: When k is even, from Lemma 3.6, let r_1 be the positive real root of the characteristic equation and r_2 be the negative real root. The $k-2$ complex roots can be paired as conjugates. From Corollary 4.3, if r and $\bar{r}$ are such a pair, then $h(r)$ and $h(\bar{r})$ are also conjugates. From equation (4), it follows that $f(x)$ can be expressed using the terms $h(r_1)r_1^x$ and $h(r_2)r_2^x$, and $(k-2)/2$ terms of the form $h(r)r^x + h(\bar{r})\bar{r}^x$.

Suppose that $h(r) = l_1 + il_2$ and $r = l_3 + il_4$. We can show that

$$h(r)r^x + h(\bar{r})\bar{r}^x = 2vw^x \cos(\theta + \gamma x),$$

where

$$v = \sqrt{l_1^2 + l_2^2} \quad \text{and} \quad w = \sqrt{l_3^2 + l_4^2}.$$

Let $\operatorname{sgn}(x)$ equal 1 if $x \ge 0$, and equal -1 if $x < 0$. We have $\theta = \operatorname{sgn}(l_2)\arccos(l_1 / v)$ and $\gamma = \operatorname{sgn}(l_4)\arccos(l_3 / w)$. We assume that, for all $x{:}-1 \le x \le 1$, $0 \le \arccos x \le \pi$. The case when k is odd is similar. □

5. SOLUTIONS

Substituting expressions for the coefficients C_i from equation (8) into equation (4) gives the unique general solution to equation (1) when the domain of f is restricted to the integers and the k initial values are known. This has been the usual application of Binet formulas.

Our solution seems to be more general than those considered previously (see [10], [13], [18], and [20]). It can also be used to find the general solutions of all homogeneous linear difference equations whose characteristic equations have simple roots.

The new generalization of the Fibonacci sequence we present defines the domain of f to be the reals. This introduces some additional questions. We first consider the solution of equation (1) when k initial values are given, and then where there is a given initial function.

6. USING THE INITIAL VALUES

6.1 Direct Solutions

Direct solutions use equation (7), $f(x) = \sum_{1\le i\le k} C_i r_i^x$. It is not difficult to show that, if $x \in \Re$ rather than the integers, then f is a solution to equation (1), as required.

The coefficients C_i can be computed following Corollary 4.3 or equation (11) by using the k initial values and k roots r_i of the characteristic equation.

From Lemma 4.4 we have that, when k is odd, and the initial values are real, then f is a real-valued function for all $x \in \Re$. When k is even and the initial values are real, the image of f can be complex when x is not an integer. This arises from the term r_2^x because $r_2 < 0$. This term can be written $(\cos(\pi x) + i\sin(\pi x))(-r_2)^x$. We can show that the real part of f is

$$h(r_1)r_1^x + h(r_2)\cos(\pi x)(-r_2)^x + \sum_{1\le i\le \frac{k}{2}-1} 2v_i w_i^x \cos(\theta_i + \gamma_i x).$$

The imaginary part of f is $h(r_2)\sin(\pi x)(-r_2)^x$. The real and imaginary parts of f individually satisfy equation (1). The real part has the same initial values as f, but the imaginary part is zero when x is an integer.

More generally, when k is even, we can replace $h(r_2)r_2^x$ with $h(r_2)m(x)(-r_2)^x$, where m is any continuous function that satisfies $m(x+1) = -m(x)$ for all $x \in \Re$, and $m(x) = (-1)^x$ when x is an integer. This family of solutions satisfies equation (1).

6.2 Laplace Transform Method

Another approach we use for finding solutions to equation (1) is based on the exponential generating function

$$G(x) = \sum_{0\le l} \frac{f(l)x^l}{l!}, \tag{12}$$

where the function f is a solution to equation (1) for a given k, where $k > 1$. First, we solve the differential equation

$$G^{(k)}(x) = \sum_{0\le i<k} G^{(i)}(x), \tag{13}$$

where $G(0) = f(0)$, $G^{(1)}(0) = f(1), \ldots, G^{(k-1)}(0) = f(k-1)$. This is done by means of the Laplace Transform [5].

We then use this solution to find an expression for $G^{(n)}(0)$, where n is a nonnegative integer. Finally, we replace the variable n by a variable $x \in \Re$, and find $f(x) = G^{(x)}(0)$.

6.2.1 Fibonacci Function

We apply the method of Section 6.2 in the case $k = 2$. The Laplace Transform of $G^{(2)}(x) = G^{(1)}(x) + G(x)$ yields

$$\bar{x} = \frac{f(0)s + f(1) - f(0)}{s^2 - s - 1} = \frac{K_1}{s - r_1} + \frac{K_2}{s - r_2}.$$

The constants K_1 and K_2 can be found by solving the following system:

$$\begin{bmatrix} 1 & 1 \\ -r_2 & -r_1 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} f(0) \\ f(1) - f(0) \end{bmatrix}.$$

This system is equivalent to that discussed in Section 4 above when $k = 2$. The solution is

$$K_1 = f(0) - K_2,$$
$$K_2 = \frac{f(1) - f(0) + r_2 f(0)}{r_2 - r_1}.$$

Applying the inverse Laplace transform yields the same result as the direct method:

$$f(x) = \frac{(f(1) - f(0)r_2)r_1^x - (f(1) - f(0)r_1)r_2^x}{\sqrt{5}}.$$

A special case occurs when $f(0) = 2$ and $f(1) = 1$. The x^{th} Lucas number L_x equals $f(x)$ when x is an integer. We call this function $L(x)$: $L(x) = r_1^x + r_2^x$. If we call $F(x)$ the solution to equation (1), where $f(0) = 0$ and $f(1) = 1$, it is not difficult to show that $L(x) = F(x-1) + F(x+1)$, $r_1^x = \frac{L(x) + \sqrt{5}F(x)}{2}$, and $(-1)^x = \frac{L^2(x) - 5F^2(x)}{4}$ for all $x \in \Re$.

6.2.2 Tribonacci Function

The Laplace Transform of equation (13) when $k = 3$ is

$$\begin{aligned} \bar{x} &= \frac{s^2 f(0) + (f(1) - f(0))s + f(2) - f(1) - f(0)}{s^3 - s^2 - s - 1} \\ &= \frac{K_1}{s - r_1} + \frac{K_2 s + K_3}{s^2 + s(r_1 - 1) + \frac{1}{r_1}}. \end{aligned}$$

The constants K_1, K_2, and K_3 can be found by solving the following system:

$$\begin{bmatrix} 1 & 1 & 0 \\ r_1 - 1 & -r_1 & 1 \\ 1/r_1 & 0 & -r_1 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} f(0) \\ f(1) - f(0) \\ f(2) - f(1) - f(0) \end{bmatrix}.$$

We have

$$K_1 = \frac{f(0)}{r_1(r_1-1)(3r_1+1)} + \frac{f(1)}{3r_1+1} + \frac{f(2)}{(r_1-1)(3r_1+1)},$$
$$K_2 = f(0) - K_1,$$
$$K_3 = \frac{K_1 - r_1(f(2)-f(1)-f(0))}{r_1^2}.$$

Hence

$$\bar{x} = \frac{K_1}{s-r_1} + \frac{K_2\left(s+\left(\frac{r_1-1}{2}\right)\right)}{\left(s+\left(\frac{r_1-1}{2}\right)\right)^2\left(\frac{1}{r_1}-\left(\frac{r_1-1}{2}\right)^2\right)} + \frac{K_3 - K_2\left(\frac{r_1-1}{2}\right)}{\left(s+\left(\frac{r_1-1}{2}\right)\right)^2 + \left(\frac{1}{r_1}-\left(\frac{r_1-1}{2}\right)^2\right)}.$$

Applying the inverse Laplace Transform gives

$$G(x) = K_1 e^{r_1 x} + K_2 e^{\left(\frac{1-r_1}{2}\right)x} \cos\sqrt{\tfrac{1}{r_1} - \left(\tfrac{r_1-1}{2}\right)^2}\,x + \frac{K_2\left(\frac{1-r_1}{2}\right)+K_3}{\sqrt{\frac{1}{r_1}-\left(\frac{r_1-1}{2}\right)^2}} e^{\left(\frac{1-r_1}{2}\right)x} \sin\sqrt{\tfrac{1}{r_1}-\left(\tfrac{r_1-1}{2}\right)^2}\,x.$$

We now use the observation that the n^{th} derivative of $e^{k_1 x}\cos(k_2 x)$ at $x = 0$, where k_1 and k_2 are constants, is $l^n \cos(\theta n)$, where $l = \sqrt{k_1^2 + k_2^2}$ and $\theta = \operatorname{sgn}(k_2)\arccos(k_1 / l)$. Similarly, the n^{th} derivative of $e^{k_1 x}\sin(k_2 x)$ at $x = 0$ is $l^n \sin(\theta n)$. We obtain

$$f(x) = K_1 r_1^x + K_2 r_1^{-x/2}\cos(\theta x) + r_1^{-x/2}\left(\frac{K_2\left(\frac{1-r_1}{2}\right)+K_3}{\sqrt{\frac{1}{r_1}-\left(\frac{r_1-1}{2}\right)^2}}\right)\sin(\theta x). \tag{14}$$

The angle θ is $\arccos\left(\frac{1-r_1}{2}\sqrt{r_1}\right)$. We can verify by induction on x that equation (14) is a solution when $k = 3$, that $C_1 = K_1$, and that this is the same function as that found by the direct method. It is interesting to note that, unlike the direct one, this solution does not use the complex roots.

6.2.3 Tetranacci Function

We shall use the method of solution of Section 6.2 when $k = 4$. For brevity, we define $V_0 = f(0)$, $V_1 = f(1) - f(0)$, $V_2 = f(2) - f(1) - f(0)$, and $V_3 = f(3) - f(2) - f(1) - f(0)$. The Laplace Transform of (13) in this case is

$$\bar{x} = \frac{V_0 s^3 + V_1 s^2 + V_2 s + V_3}{s^4 - s^3 - s^2 - s - 1}.$$

This is equivalent to

$$\bar{x} = \frac{V_0 s^3 + V_1 s^2 + V_2 s + V_3}{(s-r_1)(s-r_2)(s^2 + (r_1+r_2-1)s + r_1^2 + r_2^2 - r_1 - r_2 + r_1 r_2 - 1)}.$$

We have

$$\bar{x} = \frac{K_1}{s-r_1} + \frac{K_2}{s-r_2} + \frac{K_3 s + K_4}{s^2 + (r_1+r_2-1)s + r_1^2 + r_2^2 - r_1 - r_2 + r_1 r_2 - 1},$$

where $K_1, K_2, K_3,$ and K_4 are constants. They can be found by solving the following system:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ r_1-1 & r_2-1 & -r_1-r_2 & 1 \\ r_1^2-r_1-1 & r_2^2-r_2-1 & r_1r_2 & -r_1-r_2 \\ 1/r_1 & 1/r_2 & 0 & r_1r_2 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{bmatrix} = \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

After finding the constants, and continuing with the other steps of the method of solution, we obtain the following solution which can be verified by induction:

$$f(x) = K_1r_1^x + K_2r_2^x + y^{x/2}(K_3\cos(\theta x) + K_5\sin(\theta x)),$$

where

$$\begin{aligned}
y &= r_1^2 - r_1 + r_2^2 - r_2 + r_1r_2 - 1, \\
\theta &= \arccos\frac{1-r_1-r_2}{2\sqrt{y}}, \\
K_1 &= f(0) - K_2 - K_3, \\
K_2 &= \frac{f(1) - r_1 f(0) - K_4 - (1-2r_1-r_2)K_3}{r_2 - r_1}, \\
K_3 &= \frac{f(2) + r_1r_2 f(0) - (r_1+r_2)f(1) - (1-2r_1-2r_2)K_4}{r_1^2 - 2r_1 + r_2^2 - 2r_2 + 4r_1r_2 + 2}, \\
K_4 &= \frac{f(3) - \left(1 - \frac{1}{r_1r_2} - (r_1+r_2)K_7\right)f(1) - (1+K_7)f(2) - K_6}{r_1r_2 + \frac{1}{r_1r_2} - (1-2r_1-2r_2)K_7}, \\
K_5 &= \frac{K_4 - \left(\frac{r_1+r_2-1}{2}\right)K_3}{\sqrt{y - \left(\frac{r_1+r_2-1}{2}\right)^2}}, \\
K_6 &= \left(1 + \frac{1}{r_1} + \frac{1}{r_2} + r_1r_2K_7\right)f(0), \\
K_7 &= \frac{1-2r_1-2r_2}{(r_1^2 + r_2^2 + 4r_1r_2 - 2r_1 - 2r_2 + 2)r_1r_2}.
\end{aligned}$$

Lemma 6.1: $f(x)$ is symmetric in r_1 and r_2.

Proof: It is easy to check that $y^{x/2}(K_3\cos(\theta x) + K_5\sin(\theta x))$ is symmetric in r_1 and r_2 because y, θ, and K_3 to K_7 are symmetric. Now K_1 is equal to

$$\frac{f(1) - r_2 f(0) - K_4 - (1-2r_2-r_1)K_3}{r_1 - r_2}.$$

This is K_2 with r_1 and r_2 interchanged. Hence, $K_1r_1^x + K_2r_2^x$ is also symmetric. □

This solution is also extensionally equivalent to the one found by the direct method, and $C_1 = K_1$, and $C_2 = K_2$. Again we see that it is not necessary to find the complex roots.

Solutions similar to this one, and the ones in Sections 6.2.2 and 6.2.1 above have appeared previously (see [21], [22], [23]) but without the preceding derivations. The method of solution described in Section 6.2 above can also be applied when the roots are expressed numerically.

7. USING THE INITIAL FUNCTION

If $k \geq 2$, then given an initial function f whose domain is the interval $[0, k)$, we can compute every value of the k-step function $f(x)$ where $x \in \Re$. To do this, we define a function F_i. This is a k^{th}-order function on the integers that satisfies equation (1) and whose initial values are

$$F_i(x) = \begin{cases} 0 & \text{if } x \neq i, \\ 1 & \text{if } x = i, \end{cases}$$

where $0 \leq i,\ x \leq k-1$. In general,

$$f(l+\varepsilon) = \sum_{0 \leq i < k} f(i+\varepsilon) F_i(l), \tag{15}$$

where l is an integer, $x = l + \varepsilon$, and $\varepsilon \in [0, 1)$. We can show by induction that

$$F_i(l) = \sum_{0 \leq j \leq i} F_0(l-j). \tag{16}$$

Equation (15) can thus be written as

$$f(l+\varepsilon) = \sum_{0 \leq i < k} f(i+\varepsilon) \sum_{0 \leq j \leq i} F_0(l-j). \tag{17}$$

Equation (17) shows that f can be defined on the real numbers in terms of the initial function and the k-step function F_0 whose domain is the integers. It is not unique. For example, from equation (16) we have, for a fixed k, that

$$F_{k-1}(l-1) = \sum_{0 \leq j \leq k-1} F_0(l-j-1),$$

i.e., $F_{k-1}(l-1) = F_0(l)$. It follows that

$$f(l+\varepsilon) = \sum_{0 \leq i < k} f(i+\varepsilon) \sum_{0 \leq j \leq i} F_{k-1}(l-j-1). \tag{18}$$

Now, from equation (11), the coefficients of equation (7), for F_{k-1}, are given by

$$h(y) = \frac{y^k - 1}{y^{k-1}(y^{k+1} - 1)}.$$

On substitution into equation (18), we have

$$f(l+\varepsilon) = \sum_{0 \leq i < k} f(i+\varepsilon) \sum_{0 \leq j \leq i} \sum_{1 \leq v \leq k} \frac{(r_v^k - 1) r_v^{l-k-j}}{r_v^{k+1} - 1},$$

where the r_v are the roots of the characteristic equation.

ACKNOWLEDGMENTS

I thank Robert Low and Brendan McKay for discussions about this work. David Boyd told me of the proof of the irreducibility of the characteristic equation. Henry Gould told me about the provenance of Identity 1.120. This research was supported in part by Christ Church, Oxford, and Oxford University Computing Laboratory.

REFERENCES

1. M. J. Bertin, A. Decomps-Guilloux, M. Grandet-Hugot, M. Pathiaux-Delefosse, & J. P. Schreiber. *Pisot and Salem Numbers*. Basel: Birkhauser, 1992.
2. R. M. Burstall & J. Darlington. "A Transformation System for Developing Recursive Programs." *Journal of the ACM* **24.1** (1977):44-67.
3. W. Bosma, J. Cannon, C. Playoust, & A. Steel. *Solving Problems with Magma*. Sydney: Computational Algebra Group, University of Sydney, 1996.
4. B. W. Char, K. O. Geddes, G. H. Gonnet, B. L. Leong, M. B. Monagan, & S. M. Watt. *Maple V Language Reference Manual*. Berlin: Springer, 1991.
5. G. Doetsch. *Introduction to the Theory and Application of the Laplace Transformation*. Tr. Walter Nader. Berlin: Springer, 1974.
6. C. V. Durrell & A. Robson. *Advanced Algebra*. Vols. I-III. London: G. Bell and Sons, 1947.
7. H. M. Edwards. *Galois Theory*. Graduate Texts in Mathematics 101. Berlin: Springer, 1984.
8. H. W. Gould. "Some Generalizations of Vandermonde's Convolution." *Amer. Math. Monthly* **63** (1956):84-91.
9. H. W. Gould. *Combinatorial Identities: A Standardized Set of Tables Listing 500 Binomial Coefficient Summations*. Morgantown: Morgantown Printing and Binding, 1972.
10. A. F. Horadam. "A Generalized Fibonacci Sequence." *Amer. Math. Monthly* **68** (1961):455-59.
11. S. Lang. *Algebra*. 3rd ed. Reading, Mass.: Addison-Wesley, 1993.
12. C. L. Liu. *Introduction to Combinatorial Mathematics*. New York: McGraw-Hill, 1968.
13. E. P. Miles, Jr. "Generalized Fibonacci Numbers and Associated Matrices." *Amer. Math. Monthly* **67** (1960):745-52.
14. M. D. Miller. "On Generalized Fibonacci Numbers." *Amer. Math. Monthly* **78** (1971):1008-09.
15. G. Pólya & G. Szegö. *Problems and Theorems in Analysis*. Vol. 1. Die Grundlehren der mathematischen Wissenschaften, Band 193. Tr. from the 1925 German ed. by Dorothee Aeppli. Berlin: Springer, 1972.
16. A. P. Prudnikov, Yu. A. Brychkov, & O. I. Marichev. *Integrals and Series*. Vol. 1: *Elementary Functions*. 2nd printing. Tr. N. M. Queen. New York: Gordon and Breach Science, 1986.
17. W. R. Spickerman, R. L. Creech, & R. N. Joyner. "On the $(3, F)$ Generalizations of the Fibonacci Sequence." *The Fibonacci Quarterly* **33.1** (1995):9-12.
18. W. R. Spickerman & R. N. Joyner. "Binet's Formula for the Recursive Sequence of Order k." *The Fibonacci Quarterly* **22.4** (1984):327-31.
19. A. Valibouze. "Computation of the Galois Groups of the Resolvent Factors for the Direct and Inverse Galois Problems." In *Applied Algebra, Algebraic Algorithms and Error-Correcting Codes*, pp. 456-68. 11th International Symposium. Ed. Gérard Cohn, Marc Giusti, & Teo Mora. Lecture Notes in Computer Science **948**. Berlin: Springer, 1995.
20. M. E. Waddill. "The Tetranacci Sequence and Its Generalizations." *The Fibonacci Quarterly* **30.1** (1992):9-20.
21. D. A. Wolfram. "Generalizing Fibonacci's Sequence." *Telicom: Journal of the International Society for Philosophical Enquiry* **X.22** (1992):25.
22. D. A. Wolfram. "The Three Step Function. *Ibid.* **XI.4** (1993):48.
23. D. A. Wolfram. "The Four Step Function. *Ibid.* **XI.7** (1993):48.

AMS Classification Numbers: 11B39, 39A10