A NOTE ON STIRLING NUMBERS OF THE SECOND KIND

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1. INTRODUCTION

In [3], Todorov proved a theorem related to the explicit expression for Stirling numbers of the second kind, $S(n, m)$, in a very complicated way. In this paper, we shall prove that this result is a consequence of the well-known representation of the Stirling numbers of the second kind.

Starting from the rational generating function for Stirling numbers of the second kind,

$$\frac{t^m}{(1-t)(1-2t)\cdots(1-mt)} = \sum_{n=m}^{\infty} S(n, m)t^n, \tag{1}$$

we find that the left side of (1) is identical to

$$\begin{aligned} &t^m(1+t+t^2+\cdots)(1+2t+2^2t^2+\cdots)\cdots(1+mt+m^2t^2+\cdots) \\ &= \sum_{n=m}^{\infty}\left(\sum_{k_1+k_2+\cdots+k_m=n-m} 1^{k_1}2^{k_2}\cdots m^{k_m}\right)t^m. \end{aligned} \tag{2}$$

If we identify coefficients of t^n from equations (1) and (2), we get (see Aigner [1] or Comtet [2]):

$$S(n, m) = \sum_{k_1+k_2+\cdots+k_m=n-m} 1^{k_1}2^{k_2}\cdots m^{k_m}.$$

This formula is identical to

$$S(n, m) = \sum_{1\le i_1\le i_2\le\cdots\le i_{n-m}\le m} i_1 i_2 \cdots i_{n-m}. \tag{3}$$

In this paper, we prove that Todorov's expression for Stirling numbers of the second kind (see [3]) is a simple consequence of the representation (3).

1. THE MAIN RESULT

Let us take, in (3), the change of indices in the following way:

$$i_s = j_s - s \quad (s = 1, 2, \ldots, k). \tag{4}$$

Then, from $1 \le i_1 \le i_2$, we have $2 \le i_1 + 1 \le i_2 + 1$, i.e., $2 \le j_1 \le j_2 - 1$. Similarly, from

$$(\forall s \in \{1, 2, \ldots, k\}), \quad s \le i_{s-1} + s - 1 \le i_s + s - 1,$$

using (4), we get

$$s \le j_{s-1} \le j_s - 1 \quad (s = 2, \ldots, k).$$

For $k = n - m$, we obtain $k + 1 \le j_k - (n - m) \le m$, i.e., $k + 1 \le j_k \le n$. So, the sum on the right side of the equality (3) is identical to

$$S(n,m) = \sum_{j_k=k+1}^{n} \sum_{j_{k-1}=k}^{j_k-1} \cdots \sum_{j_2=3}^{j_3-1} \sum_{j_1=2}^{j_2-1} (j_k - k)(j_{k-1} - k + 1) \cdots (j_1 - 1), \tag{5}$$

which is the result from [3].

Example: We use $n=6$, $m=3$, and $k=n-m=3$. Following the change of indices from the equality (4), we get $i_1 = j_1 - 1$, $i_2 = j_2 - 2$, and $i_3 = j_3 - 3$. Then, from $1 \le i_1 \le i_2 \le i_3 \le 3$, we have $2 \le j_1 \le j_2 - 1$, $3 \le j_2 \le j_3 - 1$, and $4 \le j_3 - 3 \le 3$, i.e., $4 \le j_3 \le 6$.

After these transformations, from formula (3) it follows that

$$\begin{aligned} S(6,3) = 90 &= \sum_{1 \le i_1 \le i_2 \le i_3 \le 3} i_1 i_2 i_3 \\ &= 1\cdot1\cdot1 + 1\cdot1\cdot2 + 1\cdot1\cdot3 + 1\cdot2\cdot2 + 1\cdot2\cdot3 + 1\cdot3\cdot3 + 2\cdot2\cdot2 + 2\cdot2\cdot3 + 2\cdot3\cdot3 + 3\cdot3\cdot3 \\ &= 1\cdot1\cdot1 + 2\cdot(1\cdot1 + 2\cdot1 + 2\cdot2) + 3\cdot(1\cdot1 + 2\cdot1 + 2\cdot2 + 3\cdot1 + 3\cdot2 + 3\cdot3) \\ &= \sum_{j_3=4}^{6} \sum_{j_2=3}^{j_3-1} \sum_{j_1=2}^{j_2-1} (j_3 - 3)(j_2 - 2)(j_1 - 1), \end{aligned}$$

which is formula (5), where we use $n=6$, $m=3$, and $k=n-m=3$.

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