THE ZECKENDORF DECOMPOSITION OF CERTAIN FIBONACCI-LUCAS PRODUCTS

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1. INTRODUCTION

The decomposition of any positive integer N as a sum of *positive-subscripted, distinct, non-consecutive* Fibonacci numbers F_k is commonly referred to as the *Zeckendorf decomposition* of N (ZD of N, in brief) [10]. This decomposition is always possible and, apart from the equivalent use of F_1 instead of F_2 (or vice-versa), is *unique* [8].

In the past years sequences of integers $\{a/b\}$, where a and b are certain Fibonacci and/or Lucas numbers (L_k), have been investigated from the point of view of the ZD of their terms (e.g., see [3], [4], [5]). The aim of this paper is to extend these studies to sequences $\{ab\}$. More precisely, in Section 2 we establish the ZD of mF_hF_k and mL_hL_k, with h and k arbitrary positive integers (possibly subject to some trivial restrictions), for the first few positive values of the integer m; the ZD of $F_hL_k, F_h^2L_k$, and $F_hL_k^2$ are also found. In Section 3, after some brief considerations on the ZD of nF_n, we analyze certain Fibonacci-Lucas products that emerge from particular choices of n.

All the identities presented in this paper have been established by proving conjectures based on behavior that became apparent through the study of early cases of h, k, and n. These conjectures were made with the aid of a multi-precision program including the generation of large-subscripted Fibonacci numbers. On the other hand, once the identities were conjectured, their proofs appeared to be rather easy and similar to one another so that, to save space, we confine ourselves to proving but a few among them; this is done in Section 4. Section 5 provides a glimpse of possible further investigations. It is worth mentioning that formula (1.4) of [4], namely,

$$\sum_{j=1}^{h} M_{rj+t} = \frac{M_{r(h+1)+t} - (-1)^r M_{rh+t} - M_{r+t} + (-1)^r M_t}{L_r - (-1)^r - 1} \tag{1.1}$$

(here, M_r stands for either F_r or L_r), plays a crucial role throughout the proofs.

2. THE ZD OF SOME FIBONACCI-LUCAS PRODUCTS

General Remarks

(a) The identities established in this section involve two integral parameters (namely, k and n) and, in most cases, are *valid* for all *positive* values of them. Sometimes they hold also for $n = 0$.

In general, some restrictions have to be imposed on k and n to obtain the ZD (as defined in Section 1) of the quantities on their left-hand sides.

(b) The number of addends in the ZD of the quantities under study depends only on the integer k. In some cases, it is even independent of k, thus assuming a constant value. In light of [2] and [1] (see also [6], p. 147), this fact is not very surprising.

(c) The usual convention that a sum vanishes whenever the upper range indicator is less than the lower one is adopted here. For brevity, we use the notation $F_{a\pm b} = F_{a+b} + F_{a-b}$.

2.1 Fibonacci Products

Proposition 1:

$$F_k F_{k+n} = \begin{cases} \sum_{j=1}^{k/2} F_{4j+n-2} & (k \text{ even}), \\ F_{n+1} + \sum_{j=1}^{(k-1)/2} F_{4j+n} & (k \text{ odd}). \end{cases} \tag{2.1}$$

Remark 1: Expression (2.1) works for $n = 0$ as well. In this case it yields the same result as that obtained by letting $s = 1$ in formulas (2.2) and (2.3) of [5].

Proposition 2:

$$2F_k F_{k+n} = \begin{cases} F_n + F_{2k+n-1} + \sum_{j=1}^{(k-2)/2} F_{4j+n+1} & (k \text{ even}), \\ F_{n+1} + F_{2k+n-1} + \sum_{j=1}^{(k-1)/2} F_{4j+n-1} & (k \geq 3, \text{ odd}). \end{cases} \tag{2.2}$$

Proposition 3: If $n \geq 2$, then

$$3F_k F_{k+n} = \begin{cases} F_n + F_{n+2} + F_{2k+n-3} + F_{2k+n} + \sum_{j=1}^{(k-4)/2} F_{4j+n+3} & (k \geq 4, \text{ even}), \\ F_{n-1} + F_{n+2} + F_{2k+n-3} + F_{2k+n} + \sum_{j=1}^{(k-3)/2} F_{4j+n+1} & (k \geq 5, \text{ odd}). \end{cases} \tag{2.3}$$

Proposition 4: If $n \geq 3$, then

$$4F_k F_{k+n} = \begin{cases} F_{n-2} + F_{n+1} + F_{n+3} + F_{2k+n+1} + \sum_{j=1}^{(k-4)/2} F_{4j+n+4} & (k \geq 4, \text{ even}), \\ F_{n-1} + F_{n+3} + F_{2k+n+1} + \sum_{j=1}^{(k-3)/2} F_{4j+n+2} & (k \geq 3, \text{ odd}). \end{cases} \tag{2.4}$$

Proposition 5: For $k, n \geq 3$, the ZD of $5F_k F_{k+n}$ is given by the right-hand side of (2.5) once the parity of k has been reversed. This fact becomes apparent upon inspection of (1.6) of [4].

2.2 Lucas Products

Proposition 6: If $n \geq 3$, then

$$L_k L_{k+n} = \begin{cases} F_{n-1} + F_{n+1} + F_{2k+n\pm1} & (k \text{ even}), \\ F_{n-2} + F_{n+1} + F_{2k+n+1} + \sum_{j=1}^{k-2} F_{2j+n+2} & (k \geq 3, \text{ odd}) \end{cases} \tag{2.5}$$

Remark 2: The ZD of L_k^2 is given by (4.2) and (4.3) of [5]. The decomposition (2.5) (k even) follows immediately from (17a) of [9].

Proposition 7: If $n \geq 5$, then

$$2L_k L_{k+n} = \begin{cases} F_{n\pm3} + F_{2k+n\pm3} & (k \geq 4, \text{ even}), \\ F_{n-4} + F_{2k+n+3} + \sum_{j=1}^{3} F_{2j+n-3} + \sum_{j=1}^{k-4} F_{2j+n+4} & (k \geq 5, \text{ odd}). \end{cases} \tag{2.6}$$

Proposition 8: If $n \geq 5$, then

$$3L_k L_{k+n} = \begin{cases} \sum_{j=1}^{4} (F_{2j+n-5} + F_{2j+2k+n-5}) & (k \geq 4, \text{ even}), \\ F_{n-4} + F_{n+3} + \sum_{j=1}^{3} F_{2j+2k+n-3} + \sum_{j=1}^{k-4} F_{2j+n+4} & (k \geq 5, \text{ odd}). \end{cases} \tag{2.7}$$

Proposition 9: If $n \geq 6$, then

$$4L_k L_{k+n} = \begin{cases} \sum_{j=1}^{4} (F_{3j+n-8} + F_{3j+2k+n-8}) & (k \geq 6, \text{ even}), \\ F_{n-4} + F_{n-2} + F_{n+1} + \sum_{j=1}^{3} F_{3j+2k+n-5} + \sum_{j=1}^{k-5} F_{2j+n+4} & (k \geq 5, \text{ odd}). \end{cases} \tag{2.8}$$

2.3 Mixed Products

That $F_k L_k = F_{2k}$ is a well-known fact (e.g., see I_7 of [7]).

Proposition 10:

$$F_k L_{k+n} = \begin{cases} \sum_{j=1}^{k} F_{2j+n-1} & (k \text{ even}), \\ F_n + F_{2k+n} & (k \text{ odd}, nk \neq 1), \end{cases} \tag{2.9}$$

$$L_k F_{k+n} = \begin{cases} F_n + F_{2k+n} & (k \text{ even}), \\ \sum_{j=1}^{k} F_{2j+n-1} & (k \text{ odd}). \end{cases} \tag{2.10}$$

Proposition 11: If $n \geq k$, then

$$F_k^2 L_{k+n} = \begin{cases} F_{n\pm k+1} + F_{n+k-2} + \sum_{j=1}^{(k-2)/2} (F_{4j+n-k} + F_{4j+n+k+2}) & (k \geq 4, \text{ even}), \\ F_{n\pm k+1} + \sum_{j=1}^{(k-1)/2} (F_{4j+n-k} + F_{4j+n+k}) & (k \geq 3, \text{ odd}). \end{cases} \tag{2.11}$$

Proposition 12: If $n \geq k+1$, then

$$L_k^2 F_{k+n} = \begin{cases} F_{n-k} + F_{n+k-2} + F_{n+k+1} + F_{n+3k} & (k \geq 4, \text{ even}), \\ F_{n-k} + F_{n+k-1} + \sum_{j=1}^{k-1} F_{2j+n+k+1} & (k \geq 3, \text{ odd}). \end{cases} \tag{2.12}$$

Remark 3: The ZD of $L_2^2 F_{2+n}$ is given by (2.12) above for $n \geq 4$. The decompositions (2.9) (k odd) and (2.10) (k even) follow immediately from (15b) and (15a) of [9], respectively. Further, it is worth mentioning that (30) and (31) of [9] follow by letting $n = 1$ in (2.10).

3. ON THE ZD OF nF_n

A brief study of the ZD of nF_n, beyond being worth undertaking *per se*, allows us to extend the results presented in Section 2 by considering some interesting Fibonacci-Lucas products that result from particular choices of n.

Definitions:

(1) Let $f(N)$ denote the number of addends in the ZD of N.
(2) Let $Q(n)$ denote nF_n.
(3) If F_n is in the ZD of $Q(n)$, then n is said to possess the property $\mathcal{P}$ (n has $\mathcal{P}$, in brief).

We are struck by two particular aspects of the ZD of $Q(n)$ that emerge from a computer experiment carried out for $1 \leq n \leq 10000$. Namely, we observe that

(i) $f[Q(n)]$ is relatively small,
(ii) If n has $\mathcal{P}$, then $n+1$ and $n+2$ have not, whereas either $n+3$ or $n+4$ has.

The numerical evidence leads us to offer the following conjectures.

Conjecture 1: The ratio of the number of naturals not having $\mathcal{P}$ to that of those having $\mathcal{P}$ is $\alpha^2 = 1 + \alpha = 1 + (1+\sqrt{5})/2$.

Conjecture 2: If $m \leq L_{2k} - 1$, with $k \geq 0$, then mL_{2k+1} has $\mathcal{P}$.

Conjecture 3: If $m \leq L_{2k-1}$, with $k \geq 1$, then $mL_{2k} + 1$ has $\mathcal{P}$.

Note. As the final draft of the paper was being prepared, the second author and Laura Sanchis discovered what seems to be a proof of Conjecture 1. Once the details have been verified, the proof will appear in a separate paper.

As for observation (i), we state the following theorem which will be proved in Section 4.

Theorem 1: If $n \le L_{2k+1}$, then $f[Q(n)] \le 2k+1$ [cf. (3.1)].

The following further results have been establised by us.

Proposition 13 (see Conj. 2): Both L_{2k+1} and $2L_{2k+1}$ have $\mathcal{P}$. More precisely, we have

$$Q(L_{2k+1}) = \sum_{j=1}^{2k+1} F_{2j+L_{2k+1}-2(k+1)}, \tag{3.1}$$

$$Q(2L_{2k+1}) = F_{2L_{2k+1}\pm 2(k+1)} + \sum_{j=1}^{2k-1} F_{2j+2L_{2k+1}-2k}. \tag{3.2}$$

Remark 4: The property $\mathcal{P}$ becomes apparent in (3.1) and (3.2) for $j = k+1$ and k, respectively.

Proposition 14 (see Conj. 3): For $k \ge 2$, both $L_{2k}+1$ and $2L_{2k}+1$ have $\mathcal{P}$. More precisely, we have

$$Q(L_{2k}+1) = F_{L_{2k}+1} + 1 + F_{L_{2k}\pm 2k+1}, \tag{3.3}$$

$$Q(2L_{2k}+1) = F_{2L_{2k}+1} + F_{2L_{2k}\pm 2k-1} + F_{2L_{2k}\pm 2k+2}. \tag{3.4}$$

Proposition 15: For $k \ge 3$, $L_k - 3$ has $\mathcal{P}$. More precisely, we have

$$Q(L_k - 3) = \begin{cases} F_{L_k-k-3} + F_{L_k-6} + F_{L_k-3} + F_{L_k-1} + \sum_{j=1}^{(k-4)/2} F_{2j+L_k} & (k \text{ even}), \\ F_{L_k-3} + \sum_{j=1}^{(k-3)/2} (F_{2j+L_k-k-4} + F_{2j+L_k-1}) & (k \text{ odd}). \end{cases} \tag{3.5}$$

Proposition 16 [cf. (3.1)]:

$$Q(L_{2k}) = F_{L_{2k}\pm 2k} \quad \text{(from (1.5) of [4])}. \tag{3.6}$$

Proposition 17:

$$Q(F_k) = \begin{cases} \sum_{j=1}^{k/2} F_{4j+F_k-k-2} & (k \text{ even}), \\ F_{F_k-k+1} + \sum_{j=1}^{(k-1)/2} F_{4j+F_k-k} & (k \text{ odd}). \end{cases} \tag{3.7}$$

We observe that the number of addends in some of the decompositions above is independent of k. In fact, from (3.6), (3.3), and (3.4), it is seen that, if $k \ge 1$, then $f[Q(L_{2k})] = 2$ whereas, if $k \ge 2$, then $f[Q(L_{2k}+1)] = 3$ and $f[Q(2L_{2k}+1)] = 5$.

Question. Let $T \ge 1$ be an arbitrary positive integer. Does there exist at least one function $g(k)$ of k for which $f\{Q[g(k)]\} = T$ for all k greater than or equal to a certain minimum value k_0?

Let us conclude this section by showing that, if $T = 4$, then there is such a function. Namely, $g(k) = L_{2k} + 3$ will work for $k \ge k_0 = 2$.

Proposition 18: If $k \geq 2$, then

$$Q(L_{2k}+3) = F_{L_{2k}+1} + F_{L_{2k}+5} + F_{L_{2k}\pm 2k+3}. \tag{3.8}$$

4. SOME PROOFS

Proof of (2.3) (k odd): Use (1.1) to rewrite the right-hand side of (2.3) as

$$F_{n-1} + F_{n+2} + F_{2k+n-3} + F_{2k+n} + \frac{F_{2k+n-1} - F_{2k+n-5} - F_{n+5} + F_{n+1}}{5}$$

$$= 2F_{n+1} + 2F_{2k+n-1} + \frac{L_{2k+n-3} - L_{n+3}}{5} \quad \text{(from (1.5) of [4])}$$

$$= \textit{idem} + \frac{L_{n+k+(k-3)} - L_{n+k-(k-3)}}{5} = \textit{idem} + \frac{5F_{n+k}F_{k-3}}{5} \quad \text{(from (1.6) of [4])}$$

$$= 2(F_{n+k+(k-1)} + F_{n+k-(k-1)}) + F_{n+k}F_{k-3}$$

$$= 2F_{n+k}L_{k-1} + F_{n+k}F_{k-3} \quad \text{(from (1.5) of [4])}$$

$$= F_{n+k}(2L_{k-1} + F_{k-3}) = 3F_kF_{n+k}.$$

Proof of (2.8) (k even): By using (1.5) of [4], the right-hand side of (2.8) becomes

$$L_k \sum_{j=1}^{4} F_{3j+n+k-8} = L_k \frac{F_{n+k+7} + F_{n+k+4} - F_{n+k-5} - F_{n+k-8}}{4} \quad \text{[from (1.1)]}$$

$$= L_k \frac{F_{n+k+1+6} - F_{n+k+1-6} + F_{n+k-2+6} - F_{n+k-2-6}}{4}$$

$$= L_k \frac{L_{n+k+1}F_6 + L_{n+k-2}F_6}{4} \quad \text{(from (1.5) of [4])}$$

$$= 2L_k(L_{n+k+1} + L_{n+k-2}) = 2L_k(2L_{n+k}) = 4L_kL_{n+k}.$$

Proof of (2.12):

Case 1: *k* ≥ 4 is even. Rewrite the right-hand side of (2.12) as

$$F_{n+k-2} + F_{n+k+1} + F_{n+k-2k} + F_{n+k+2k} = F_{n+k-2} + F_{n+k+1} + F_{n+k}L_{2k} \quad \text{(from (1.5) of [4])}$$

$$= 2F_{n+k} + F_{n+k}L_{2k} = F_{n+k}(L_{2k}+2)$$

$$= F_{n+k}L_k^2 \quad \text{(from identity } I_{15} \text{ of [7])}.$$

Case 2: *k* ≥ 3 is odd. First, rewrite the right-hand side of (2.12) as

$$F_{n-k} + F_{n+k-1} + F_{3k+n+1} - F_{3k+n-1} - F_{n+k+3} + F_{n+k+1} \quad \text{[from (1.1)]}$$

$$= F_{n-k} + F_{n+k-1} + F_{3k+n} - F_{n+k+2},$$

then use (1.5) of [4] thrice to rewrite the expression above as

$$F_{n-k} + F_{3k+n} - 2F_{n+k} = F_{3k+n} - F_nL_k - F_{n+k}$$

$$= F_{n+2k+k} - F_{n+2k-k} - F_nL_k = F_{n+2k}L_k - F_nL_k$$

$$= (F_{n+2k} - F_n)L_k = (F_{n+k+k} - F_{n+k-k})L_k$$

$$= F_{n+k}L_kL_k = F_{n+k}L_k^2.$$

Proof of (3.2): Put $2L_{2k+1} = h$ for notational convenience, and use (1.1) to rewrite the right-hand side of (3.2) as

$$\begin{aligned}
&F_{h-2k-2} + F_{h+2k+2} + (F_{h+2k} - F_{h+2k-2} - F_{h-2k+2} + F_{h-2k})\\
&= F_{h+2k+2} + F_{h+2k-1} + F_{h-2k-2} - F_{h-2k+1} = 2F_{h+2k+1} - 2F_{h-2k-1}\\
&= 2F_h L_{2k+1} \text{ (from (1.5) of [4])}\\
&= hF_n \overset{\text{def}}{=} Q(h).
\end{aligned}$$

Proof of (3.5) (k even): Put $L_k = h$ for notational convenience, and use (1.1) to rewrite the right-hand side of (3.5) as

$$\begin{aligned}
&F_{h-k-3} + F_{h-6} + F_{h-3} + F_{h-1} + (F_{h+k-2} - F_{h+k-4} - F_{h+2} + F_h)\\
&= F_{h-k-3} + F_{h+k-3} - F_{h+1} + F_{h-6} + F_{h-3} + F_{h-1}\\
&= hF_{h-3} - F_{h+1} + F_{h-6} + F_{h-3} + F_{h-1} \text{ (from (1.5) of [4])}\\
&= hF_{h-3} - 3F_{h-3} = (h-3)F_{h-3} \overset{\text{def}}{=} Q(h-3).
\end{aligned}$$

Proof of Theorem 1: From (2.3) and (2.4) of [6], we see that

$$f[Q(n)] \le \frac{1}{2}[V(n) + U(n)] + 1,$$

where $V(n) = \lfloor \log_\alpha n \rfloor$ $(\alpha = (1+\sqrt{5})/2)$ and $U(n)$ is an *even* number defined by $L_{U(n)-1} < n \le L_{U(n)+1}$. It must be observed that $U(n)$ is defined in [6] in a slightly different way, for the authors use the initial values $L_0 = 3$ and $L_1 = 4$ for the Lucas sequence. Now, it can be proved readily that, if $n \le L_{2k+1}$, then both $V(n)$ and $U(n)$ do not exceed $2k$. This fact, along with (4.1), prove the theorem.

5. CONCLUDING COMMENTS

As can be seen from the examples presented in this section, the identities established in this paper represent only a small sample of the possibilities available to us. A thorough investigation on the ZD of $Q(n)$ seems to be worthwhile; this study will be the object of a future paper. An attempt to prove Conjectures 2 and 3 produced the following decompositions [see also (3.1)-(3.4)] the proofs of which, based on the technique shown in Section 4, are left as an exercise to the interested reader. Namely, we see that

$$Q(3L_{2k+1}) = F_{3L_{2k+1}-2k-4} + F_{3L_{2k+1}-2k\pm1} + F_{3L_{2k+1}+2k+3} + \sum_{j=1}^{2k-2} F_{2j+3L_{2k+1}-2(k-1)} \quad (k \ge 2), \tag{5.1}$$

$$Q(4L_{2k+1}) = F_{4L_{2k+1}-2k-4} + F_{4L_{2k+1}\pm2k+1} + F_{4L_{2k+1}+2k+3} + \sum_{j=1}^{2k-2} F_{2j+4L_{2k+1}-2(k-1)} \quad (k \ge 2), \tag{5.2}$$

$$Q(5L_{2k+1}) = F_{5L_{2k+1}\pm2k} + F_{5L_{2k+1}\pm2(k+2)} + \sum_{j=1}^{2k-3} F_{2j+5L_{2k+1}-2(k-1)} \quad (k \ge 2), \tag{5.3}$$

$$Q(3L_{2k}+1) = F_{3L_{2k}+1} + F_{3L_{2k}\pm2k-1} + F_{3L_{2k}\pm2k+3} \quad (k \ge 2), \tag{5.4}$$

$$Q(4L_{2k}+1) = F_{4L_{2k}+1} + F_{4L_{2k}\pm2k-1} + F_{4L_{2k}\pm2k+1} + F_{4L_{2k}\pm2k+3} \quad (k \ge 2), \tag{5.5}$$

$$Q(5L_{2k}+1) = F_{5L_{2k}+1} + F_{5L_{2k}\pm2k-3} + F_{5L_{2k}\pm2k} + F_{5L_{2k}\pm2k+4} \quad (k \ge 3). \tag{5.6}$$

Remark 5: The property $\mathcal{P}$ becomes apparent in (5.1)-(5.3) for $j = k-1$.

Moreover, we believe that also the ZD of nL_n deserves some study. A medium-range ($1 \le n \le 2000$) computer experiment led us to conjecture that F_n is not in the ZD of nL_n for $n > 2$. This experiment allowed us to observe that, if $n = F_{2k+1}$ ($k = 1, 2, 3, \ldots$), then $f(nL_n) = 2$ with only one exception in the case $k = 2$ for which $f(5L_5) = 1$. In fact, from (1.5) of [4], it can be seen immediately that

$$F_{2k+1}L_{F_{2k+1}} = F_{F_{2k+1}\pm(2k+1)}. \tag{5.7}$$

Remark 6: Letting $k = 1$ and 2 in (5.7) yields $2L_2 = F_{-1} + F_5 = F_2 + F_5$ and $5L_5 = F_0 + F_{10} = F_{10}$ (the exception), respectively.

Further, we observed that, if $n = L_{2k}$ ($k = 2, 3, 4, \ldots$), then $f(nL_n) = 4$. In fact, from identity I_8 of [7] and (1.6) of [4], it can be proved readily that

$$L_{2k}L_{L_{2k}} = F_{L_{2k}\pm 2k-1} + F_{L_{2k}\pm 2k+1}. \tag{5.8}$$

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