DISTRIBUTION OF BINOMIAL COEFFICIENTS MODULO THREE

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1. INTRODUCTION

In 1947, Fine [1] proved that almost all binomial coefficients are even. That is, if $T_2(n)$ is the number of odd binomial coefficients $\binom{m}{k}$ with $0 \le k \le m < n$, then $T_2(n) = o(n^2)$. In particular, since the total number of such binomial coefficients is $1+2+\cdots+n = \frac{1}{2}n^2 + \frac{1}{2}n$, the proportion of odd coefficients tends to 0 with n. In 1977, Harborth [3] improved this estimate to

$$.812556 n^{\log_2 3} \le T_2(n) \le n^{\log_2 3},$$

and although the best constant in the lower bound has been calculated to great accuracy [2], its exact value is still unknown. The behavior of $T_2(n)$ and its generalizations to $T_p(n)$ for prime p have also been studied by Howard [4], Singmaster [6], Stein [7], and Volodin [8]. In the following definitions, let $\binom{0}{0} = 1$. For any prime p, it is convenient to let $P = \binom{p+1}{2}$, $\theta_p = \log_p P$, and to let $S_p(n)$ denote the number of binomial coefficients $\binom{n}{r}$ that are not divisible by p. Then

$$T_p(n) = \sum_{k=0}^{n-1} S_p(k)$$

is the number of binomial coefficients in the first n rows of Pascal's triangle that are not divisible by p. It is known (see [3] and [7]) that the quotient $R_p(n) = T_p(n)/n^{\theta_p}$ is bounded above by $\alpha_p = \sup_{n\ge1} R_p(n) = 1$ and below by $\beta_p = \inf_{n\ge1} R_p(n)$. The β_p tend to $\frac{1}{2}$ with p [2], but to this point no exact values for β_p have been found.

2. THE CASE $p = 3$

Henceforth, the terms θ, R, S, T shall denote θ_3, R_3, S_3, T_3, respectively. Also, let

$$n = \sum_{i=1}^{k} a_i 3^{r_i}$$

be n's base-three representation, where each $a_i = 1$ or 2 and $r_1 > r_2 > \cdots > r_k \ge 0$. We list the first few values of $S(n)$, $T(n)$, and $R(n)$ in Table 1.

We shall confirm a conjecture of Volodin [8], namely that $\inf_{n\ge1} R(n) = 2^{\log_3 2-1} = .77428$. The fractal nature of Pascal's triangle modulo 3 implies (see [5], Cor. 2, p. 367) the following recursive formula for T:

$$T(a\cdot 3^s + b) = \frac{1}{2}a(a+1)6^s + (a+1)T(b) \quad \text{for } a = 1 \text{ or } 2,\ b < 3^s.$$

It follows by iteration that

$$T\left(\sum_{i=1}^{k} a_i 3^{r_i}\right) = \frac{1}{2}\sum_{i=1}^{k} a_i(a_1+1)\cdots(a_i+1)6^{r_i}. \tag{1}$$

n	$S(n)$	$T(n)$	$R(n)$	n	$S(n)$	$T(n)$	$R(n)$	n	$S(n)$	$T(n)$	$R(n)$
0	1	0		10	4	38	.88890	20	9	117	.88368
1	2	1	1	11	6	42	.84103	21	6	126	.87887
2	3	3	.96864	12	4	48	.83401	22	8	132	.85345
3	2	6	1	13	8	52	.79294	23	18	144	.86592
4	4	8	.83401	14	12	60	.81077	24	9	162	.87887
5	6	12	.86938	15	6	72	.86938	25	18	171	.89754
6	3	18	.96864	16	18	78	.84773	26	27	189	.93055
7	6	21	.87887	17	12	96	.94514	27	2	216	1
8	9	27	.90884	18	3	108	.96864	40	16	320	.78037
9	2	36	1	19	6	111	.91152	121	32	1936	.77630

3. MAIN RESULT

Theorem 1: The number of binomial coefficients $\binom{m}{k}$, $k \le m < n$, that are not divisible by 3 is bounded below by $2^{\log_3 2 - 1} n^{\log_3 6}$ and this bound is sharp.

Proof: Let the two sequences $\mathbf{x}$, $\mathbf{y}$ be defined by

$$x_i = 3^{r_i}\left[\frac{1}{2}a_i(a_1+1)\cdots(a_i+1)\right]^{\frac{1}{\theta}} \quad \text{and} \quad y_i = a_i\left[\frac{1}{2}a_i(a_1+1)\cdots(a_i+1)\right]^{\frac{-1}{\theta}}, \quad 1 \le i \le k.$$

We apply Hölder's inequality to the sequences $\mathbf{x}$, $\mathbf{y}$ with the conjugate exponents $\theta = \log_3 6$ and $\theta' = \log_2 6$:

$$\sum_{i=1}^{k} x_i y_i \le \left(\sum_{i=1}^{k} x_i^{\theta}\right)^{\frac{1}{\theta}} \cdot \left(\sum_{i=1}^{k} y_i^{\theta'}\right)^{\frac{1}{\theta'}},$$

$$n \le \left(\sum_{i=1}^{k}\left\{3^{r_i}\left[\frac{1}{2}a_i(a_1+1)\cdots(a_i+1)\right]^{\frac{1}{\theta}}\right\}^{\theta}\right)^{\frac{1}{\theta}} \cdot \left(\sum_{i=1}^{k}\left\{a_i\left[\frac{1}{2}a_i(a_1+1)\cdots(a_i+1)\right]^{\frac{-1}{\theta}}\right\}^{\theta'}\right)^{\frac{1}{\theta'}},$$

$$n^{\theta} \le \left(\sum_{i=1}^{k} 6^{r_i}\frac{1}{2}a_i(a_1+1)\cdots(a_i+1)\right) \cdot \left(\sum_{i=1}^{k} a_i^{\theta'}\left[\frac{1}{2}a_i(a_1+1)\cdots(a_i+1)\right]^{-\frac{\theta'}{\theta}}\right)^{\frac{\theta}{\theta'}},$$

$$R(n) \ge \frac{1}{2}\left(\sum_{i=1}^{k} a_i\left[(a_1+1)\cdots(a_i+1)\right]^{-\frac{\theta'}{\theta}}\right)^{-\frac{\theta}{\theta'}}. \tag{2}$$

Let $v = \theta' / \theta = \log_2 3 = 1.58496$ and let

$$U_k = \sum_{i=1}^{k} a_i[(a_1+1)\cdots(a_i+1)]^{-v}.$$

Note that $U_k = f_1 \circ f_2 \circ f_3 \circ \cdots \circ f_k(0)$, where

$$f_i(x) = \frac{x+a_i}{(a_i+1)^v}.$$

Each f_i is one of the two increasing functions $\frac{x+1}{3}$ or $\frac{x+2}{3^v}$ and U_k will be maximized when each a_i is chosen to maximize f_i. For a given x, we find that $\frac{x+1}{3} > \frac{x+2}{3^v}$ (i.e., $a_i = 1$) if and only if $x > .109253$. So, for $x = 0$, $f_k(0)$ is maximized when $a_k = 2$. For $i < k$, $f_i(x)$ is maximized when $a_i = 1$ since x will now be in the range of f_i and, hence, $\geq \frac{1}{3}$. Thus,

$$\begin{aligned} U_k &\leq \frac{1}{2^v} + \frac{1}{2^{2v}} + \cdots + \frac{1}{2^{(k-1)v}} + \frac{2}{2^{(k-1)v} \cdot 3^v} \\ &= \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{k-1}} + \frac{2}{3^{k-1} \cdot 3^v} = \frac{1}{2} - \frac{\frac{1}{2} - 2 \cdot 3^{-v}}{3^{k-1}} \\ &\leq \frac{1}{2} \quad \text{since } \frac{1}{2} - 2 \cdot 3^{-v} > 0. \end{aligned}$$

Hence, from (2) we have, for all n,

$$R(n) \geq \frac{1}{2}(U_k)^{-\frac{1}{v}} > \left(\frac{1}{2}\right)^{1-\frac{1}{v}} = 2^{\log_3 2 - 1},$$

whence

$$\beta_3 \geq \left(\frac{3}{2}\right)^{-\frac{1}{v}} = 2^{\log_3 2-1}.$$

We now consider numbers of the form $1+3+3^2+3^3+\cdots+3^k$. It follows from (1) that

$$\begin{aligned} R(1+3+3^2+3^3+\cdots+3^k) &= \frac{\frac{1}{2}(2\cdot 6^k + 2^2 \cdot 6^{k-1} + \cdots + 2^{k+1})}{(1+3+3^2+3^3+\cdots+3^k)^{\log_3 6}} \\ &= \frac{2^k(3^k+3^{k-1}+\cdots+1)}{\left(\frac{3^{k+1}-1}{2}\right)^{\log_3 6}} = \frac{2^k}{\left(\frac{3^{k+1}-1}{2}\right)^{\log_3 2}} = \frac{2^{k+1}}{(3^{k+1}-1)^{\log_3 2}} \cdot 2^{\log_3 2-1} \end{aligned}$$

so that $\lim_{k\to\infty} R(1+3+3^2+3^3+\cdots+3^k) = 2^{\log_3 2-1}$.

Hence, $\beta_3 \leq 2^{\log_3 2-1}$. This implies $\beta_3 = 2^{\log_3 2-1}$ and $T(n) > 2^{\log_3 2-1} n^{\log_3 6}$, the desired result. Note that n and $T(n)$ are integers, so there is strict inequality.

The proof of Theorem 1 works because the sequence $\{1, 1, 1, \ldots\}$ that minimizes $R(n)$ gives rise to sequences x_i, y_i for which equality holds in Hölder's inequality. This does not occur for $p \neq 3$, so the proof does not extend to other primes.

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Gerald E. Bergum, Editor