SOLVING LINEAR EQUATIONS USING AN OPTIMIZATION-BASED ITERATIVE SCHEME

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A system of linear equations such as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + c_1 &= 0 \\ a_{21}x_1 \qquad\quad + \cdots + a_{2n}x_n + c_2 &= 0 \\ \cdots \\ a_{n1}x_1 \qquad\quad + \cdots + a_{nn}x_n + c_n &= 0 \end{aligned} \tag{1}$$

can be solved using either direct methods such as the Gauss-Jordan procedure or iterative methods such as the Gauss-Seidel procedure. When the equation system is large, and especially when the coefficients are sparsely distributed, iterative methods are often preferred (see [1], [2]) since iterative methods for these systems are quite rapid and may be more economical in memory requirements of a computer. Iterative methods usually require a set of starting values as assumed solution. This article describes a procedure that does not require starting values. The procedure achieves convergence rapidly and can be applied to dependent systems in which there are fewer equations than variables.

Consider the case of n simultaneous equations in matrix form:

$$\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} & c_1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & c_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 1 \end{pmatrix}. \tag{2}$$

We propose to solve the system

$$\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = 0 \tag{3}$$

by minimizing a scalar objective function

$$H = f_1^2 + \cdots + f_n^2 = (f_1 \ \cdots \ f_n) \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}. \tag{4}$$

The solution of (3) is obtained when the x values are found such that $H = 0$.

Differentiation of (4) yields

$$\frac{dH}{dt} = 2(f_1 \ \cdots \ f_n) \begin{pmatrix} df_1 / dt \\ \vdots \\ df_n / dt \end{pmatrix}. \tag{5}$$

But (2) gives

$$\begin{pmatrix} df_1 / dt \\ \vdots \\ df_n / dt \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} dx_1 / dt \\ \vdots \\ dx_n / dt \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & 0 \end{pmatrix} \begin{pmatrix} dx_1 / dt \\ \vdots \\ dx_n / dt \\ dz / dt \end{pmatrix} \tag{6}$$

and, together with (2), (5) becomes

$$\frac{dH}{dt} = 2(f_1 \ \cdots \ f_n)\begin{pmatrix} a_{11} & \cdots & a_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & 0 \end{pmatrix}\begin{pmatrix} dx_1/dt \\ \vdots \\ dx_n/dt \\ dz/dt \end{pmatrix}$$

$$= 2(x_1 \ \cdots \ x_n \ \ 1)\begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{nn} \\ c_1 & \cdots & c_n \end{pmatrix}\begin{pmatrix} a_{11} & \cdots & a_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & 0 \end{pmatrix}\begin{pmatrix} dx_1/dt \\ \vdots \\ dx_n/dt \\ dz/dt \end{pmatrix}. \tag{7}$$

We now set

$$\begin{pmatrix} dx_1/dt \\ \vdots \\ dx_n/dt \\ dz/dt \end{pmatrix} = -\begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{nn} \\ 0 & \cdots & 0 \end{pmatrix}\begin{pmatrix} a_{11} & \cdots & a_{1n} & c_1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & c_n \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 1 \end{pmatrix}, \tag{8}$$

so that $dH/dt \le 0$.

By choosing a small value for Δt (for example, set it equal to the reciprocal of the Euclidean norm [1] of the matrix), one could approximate the derivatives on the left-hand side of (8) by finite differences between the $(i+1)^{\text{th}}$ and the i^{th} iterant of each of the x's (with z remaining a constant = 1), and we write

$$\begin{aligned} \frac{dx_1}{dt} &\cong \frac{\Delta x_1}{\Delta t} = \frac{x_{1,i+1} - x_{1,i}}{\Delta t}, \\ \frac{dx_n}{dt} &\cong \frac{\Delta x_n}{\Delta t} = \frac{x_{n,i+1} - x_{n,i}}{\Delta t}, \\ &\cdots \\ \frac{dz}{dt} &\cong \frac{\Delta z}{\Delta t} = \frac{z_{i+1} - z_i}{\Delta t}. \end{aligned} \tag{9}$$

Then (8) becomes

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \\ z \end{pmatrix}_{i+1} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ z \end{pmatrix}_i - \Delta t\begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{nn} \\ 0 & \cdots & 0 \end{pmatrix}\begin{pmatrix} a_{11} & \cdots & a_{1n} & c_1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & c_2 \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 1 \end{pmatrix}_i, \tag{10}$$

with $z_{i+1} = z_i = 1$.

Let

$$[B] = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} - \Delta t\begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{nn} \\ 0 & \cdots & 0 \end{pmatrix}\begin{pmatrix} a_{11} & \cdots & a_{1n} & c_1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & c_n \end{pmatrix} \tag{11}$$

and

$$[X] = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 1 \end{pmatrix}, \tag{12}$$

then (10) becomes the recursion equation

$$[X]_{i+1} = [B]\ [X]_i. \tag{13}$$

Starting with an arbitrary set of values $[X]=[X]_0$ with $z=1$ as the initial solution, we carry out the iteration

$$\begin{aligned}[X]_1 &= [B]\ [X]_0\\ [X]_2 &= [B]\ [X]_1\\ &= [B]^2\,[X]_0\end{aligned} \tag{14}$$

and so on until

$$[X]_k = [B]^k\,[X]_0.$$

Unless the set of equations is an inconsistent system, for sufficiently small Δt, which serves as an accelerating factor, $[B]^k$ will converge so that

$$[B]^k \to \begin{pmatrix} b_{11} & \cdots & b_{1n} & x_{1,s}\\ \vdots & & \vdots & \vdots\\ b_{n1} & \cdots & b_{nn} & x_{n,s}\\ 0 & \cdots & 0 & 1\end{pmatrix} \tag{15}$$

as $k\to\infty$. It follows from (13) that

$$[X]_k = \begin{pmatrix} b_{11} & \cdots & b_{1n} & x_{1,s}\\ \vdots & & \vdots & \vdots\\ b_{n1} & \cdots & b_{nn} & x_{n,s}\\ 0 & \cdots & 0 & 1\end{pmatrix}[X]_0. \tag{16}$$

If the equation system is furthermore not a dependent system, the individual elements b_{ij} will tend to zero upon convergence, and

$$[X]_k = \begin{pmatrix} x_{1,s}\\ \vdots\\ x_{n,s}\\ 1\end{pmatrix} \tag{17}$$

regardless of the initial starting value, and $[X]=[x_s]$ are obtained for the solution of (2).

Since the last column of the matrix (15) to which $[B]$ converges contains the solution of (2), it is clear that to solve a set of linear equations that does not constitute a dependent system, an assumed starting solution is not required. We only need to multiply $[B]$ of (11) upon itself repeatedly, and if the multiplication is performed by squaring the previous result, convergence is accelerated through the sequence $[B],[B]^2,[B]^4,\ldots$. Although the process diverges for an overdetermined system, for a dependent system, one can obtain a solution from (15) according to the initial starting solution vector.

To illustrate the scheme, consider the following examples.

1. Consider the system: $\begin{aligned} x+y&=-2\\ 2x+y&=-3\end{aligned}$

$$[B]=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\Delta t\begin{pmatrix}1&2\\1&1\\0&0\end{pmatrix}\begin{pmatrix}1&1&2\\2&1&3\end{pmatrix}=\begin{pmatrix}0.95&-0.03&-0.08\\-0.03&0.98&-0.05\\0&0&1\end{pmatrix},$$

for which the acceleration factor $\Delta t = 0.01$ is used. Convergence leads to

$$[B]^n \to \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

From the last column, the solution $(x, y) = (-1, -1)$ is obtained.

2. Given the dependent system: $x + y + z + 2 = 0$

$$2x + y - z + 3 = 0$$

$$[B] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \Delta t \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 0.95 & -.03 & 0.1 & -0.08 \\ -0.03 & 0.98 & 0 & -0.05 \\ 0.01 & 0 & 0.98 & 0.01 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

for which the acceleration factor Δt is again 0.01. Iteration leads to

$$[B]^n \to \begin{pmatrix} 0.2857 & -0.428 & 0.1428 & -1.142 \\ -0.428 & 0.6428 & -0.214 & -0.785 \\ 0.1428 & -0.214 & 0.0714 & -0.071 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The solution is found from (15) for any arbitrarily chosen starting value. For instance, if $x_0 = y_0 = z_0 = 0$, then, from the last column, we obtain $(x, y, z) = (-1.142, -0.785, -0.071)$. If $x_0 = 2, y_0 = 0$, and $z_0 = 1$, then $(x, y, z) = (-0.714, -1.427, 0.143)$ is obtained.

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REFERENCES

1. Curtis F. Gerald & P. O. Wheatley. *Applied Numerical Analysis*. 4th ed. New York: Addison Wesley, 1992.
2. M. L. James, G. M. Smith, & J. C. Wolford. *Applied Numerical Methods for Digital Computation*. New York: Harper Collins, 1993.

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