A CHAOTIC EXTENSION OF THE $3x+1$ FUNCTION TO $\mathbb{Z}_2[i]$

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1. INTRODUCTION

The $3x+1$ problem is most elegantly expressed in terms of iteration of the function $T:\mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$T(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

The $3x+1$ conjecture, which is generally attributed to Collatz [3], is that for every positive integer n, $T^{(k)}(n)=1$ for some k.

The function T can be extended in a natural manner to the 2-adic integers, $\mathbb{Z}_2$, and this extension has proven to be quite fruitful. In this paper, we further extend the domain of T to $\mathbb{Z}_2[i]$, hoping to increase our understanding of the problem.

We construct an extension, $\widetilde{T}$, of the $3x+1$ function, $T:\mathbb{Z}_2 \to \mathbb{Z}_2$ to the metric space $(\mathbb{Z}_2[i], D)$, as follows.

Definition 1: Let $\widetilde{T}:\mathbb{Z}_2[i] \to \mathbb{Z}_2[i]$ by

$$\widetilde{T}(\alpha) = \begin{cases} \frac{\alpha}{2} & \text{if } \alpha \in [0], \\ \frac{3\alpha+1}{2} & \text{if } \alpha \in [1], \\ \frac{3\alpha+i}{2} & \text{if } \alpha \in [i], \\ \frac{3\alpha+1+i}{2} & \text{if } \alpha \in [1+i], \end{cases}$$

where $[x]$ denotes the equivalence class of x in $\mathbb{Z}_2[i]/2\mathbb{Z}_2[i]$.

Our main results separate naturally into three areas.

First, $\widetilde{T}$ is an extension of the original function and is nontrivial in the following sense.

Theorem A:
(a) $\widetilde{T}|\mathbb{Z}_2 = T$;
(b) $\widetilde{T}$ is not conjugate to $T\times T$ via a $\mathbb{Z}_2$-module isomorphism;
(c) $\widetilde{T}$ is, however, topologically conjugate to $T\times T$.

Second, $\widetilde{T}$ preserves the salient qualities of the original function. In particular, there is a "parity vector function," Q_∞, for T which has been extremely important in understanding the nature of the problem, see [1], [2], [5], [6], [7]. We show that Q_∞ can also be extended in an analogous manner. The original parity vector function and the ex-tended parity vector function share several important properties (c.f. [5], Theorem B). We simply state the results here, saving the details for later in the paper.

Theorem B: The extended parity vector function $\widetilde{Q}_k$ is periodic with period 2^k. In addition, $\widetilde{Q}_\infty$ is an isometric homeomorphism.

Finally, T and $T \times T$ are both chaotic functions (in the sense of [4]) and thus it follows from part (c) of Theorem A that

Theorem C: $\widetilde{T}$ is chaotic.

We have constructed $\widetilde{T}$ in the hope that the results above may lead to further insight into the $3x+1$ problem and similar problems.

2. BACKGROUND AND NOTATION

In this section, we establish notation and discuss the relevant background material. Lagarias [5] describes the history of the $3x+1$ problem and gives a survey of the literature on the subject. We follow his notation.

The sequence n, $T(n)$, $T^{(2)}(n)$, $T^{(3)}(n)$, ... is called the *orbit* of n under T. Another way to state the $3x+1$ conjecture is that, for all $n \in \mathbb{Z}^+$, the orbit of n under T enters the cycle $2 \to 1 \to 2 \to 1 \to \ldots$. Since T extends naturally to the ring of 2-adic integers, $\mathbb{Z}_2$, the statement of the $3x+1$ problem is also valid on $\mathbb{Z}_2$. For brevity, we shall often refer to a 2-adic integer as simply a "2-adic." Recall that an element, a, of $\mathbb{Z}_2$ is just a formal power series of the form $\sum_{i=0}^{\infty} a_i 2^i$, where $a_i \in \{0, 1\}$. As is common, we will often abbreviate this by writing the sequence of 0s and 1s as $a_0, a_1, a_2, \ldots_2$. Note that we add the subscript 2 (to distinguish from base 10) when writing 2-adics and use an overbar to denote a repeating pattern. Note also that both $\mathbb{Z}$ and the set of rationals with odd denominators are subrings of $\mathbb{Z}_2$ and thus, for clarity, we will frequently write an integer or rational number in place of its 2-adic representation. For example, $\overline{10}_2$ denotes the 2-adic $\sum_{i=0}^{\infty} 2^{2i}$ associated with $-\frac{1}{3}$.

Define the *parity vector of length k for T of a* [5] to be the sequence given by the function $Q_k : \mathbb{Z}_2 \to \mathbb{Z}_2 / 2^k \mathbb{Z}_2$ by

$$Q_k(a) = x_0(a), x_1(a), \ldots, x_{k-1}(a),$$

where $x_i(n) \equiv T^{(i)}(n) \bmod 2$ and $x_i(n) \in \{0, 1\}$ for all $i \geq 0$. The parity vector, $Q_k(a)$, completely describes the behavior of the first k iterates of a under T. $Q_\infty(a)$ is defined in a similar manner and completely describes all iterates of a under T.

Q_k and Q_∞ have several interesting properties: Q_k is periodic with period 2^k and induces a permutation of $\mathbb{Z}_2 / 2^k \mathbb{Z}_2$, denoted $\overline{Q}_k$; Q_∞ is a continuous bijection. The proofs of these properties of Q_k and Q_∞ may be found in [5]. Both have proven to be extremely useful in the study of the $3x+1$ problem.

In this paper we extend T to the 2-adic integers adjoined with i, $\mathbb{Z}_2[i]$. We choose to extend to $\mathbb{Z}_2[i]$ because many number theoretic problems in $\mathbb{Z}$ have been solved by generalizing to the Gaussian integers $\mathbb{Z}[i]$. In keeping with this theme, we shall refer to $\mathbb{Z}_2[i]$ as the set of *Gaussian 2-adic integers* or, simply, the *Gaussian 2-adics*.

By freely associating $a + bi \in \mathbb{Z}_2[i]$ with $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ we can define the metric D on $\mathbb{Z}_2[i]$ to be the product metric on $\mathbb{Z}_2 \times \mathbb{Z}_2$ induced by the usual metric on $\mathbb{Z}_2$ which is derived from the 2-adic valuation. Addition and multiplication in $\mathbb{Z}_2[i]$ are defined in the usual manner. It is important to note that $\mathbb{Z}_2[i]$ is a commutative ring with identity, but not a field. In addition, $\mathbb{Z}_2$ is a commutative subring of $\mathbb{Z}_2[i]$ with identity and is also not a field.

3. EXTENSION TO $\mathbb{Z}_2[i]$

Since T was piecewise linear depending on equivalence in $\mathbb{Z}/2\mathbb{Z}$, our extension $\widetilde{T}$ is piecewise linear depending on equivalence in $\mathbb{Z}_2[i]/2\mathbb{Z}_2[i]=\{[0],[1],[i],[1+i]\}$.

Definition 1: Let $\widetilde{T}:\mathbb{Z}_2[i]\to\mathbb{Z}_2[i]$ by

$$\widetilde{T}(\alpha)=\begin{cases}\frac{\alpha}{2} & \text{if } \alpha\in[0],\\ \frac{3\alpha+1}{2} & \text{if } \alpha\in[1],\\ \frac{3\alpha+i}{2} & \text{if } \alpha\in[i],\\ \frac{3\alpha+1+i}{2} & \text{if } \alpha\in[1+i].\end{cases}$$

Notice that $\widetilde{T}$ resembles $T\times T$ to a great degree. It is natural to ask how $\widetilde{T}$ is different from T and $T\times T$; after all, we claim that $\widetilde{T}$ is a nontrivial extension of T. It is easily seen that $\widetilde{T}$ is an extension of T, i.e., $\widetilde{T}|\mathbb{Z}_2=T$. It is also clear that $\widetilde{T}$ is not the trivial extension $T\times T:\mathbb{Z}_2\times\mathbb{Z}_2\to\mathbb{Z}_2\times\mathbb{Z}_2$, e.g., $T\times T((1,2))=(2,1)$, while $\widetilde{T}(1+2i)=2+3i$. What is surprising, though, is that $\widetilde{T}$ and $T\times T$ are not conjugate via a $\mathbb{Z}_2$-module isomorphism, as we show below. (However, they are topologically conjugate, as we shall see in Section 6.)

In order to show this, we must formalize our association between elements of $\mathbb{Z}_2[i]$ and $\mathbb{Z}_2\times\mathbb{Z}_2$. Define a continuous bijection $B:\mathbb{Z}_2[i]\to\mathbb{Z}_2\times\mathbb{Z}_2$ by $B(a+bi)=(a,b)$. Let $\hat{T}:\mathbb{Z}_2\times\mathbb{Z}_2\to\mathbb{Z}_2\times\mathbb{Z}_2$ by $\hat{T}=B\circ\widetilde{T}\circ B^{-1}$.

Theorem 1: There is no $\mathbb{Z}_2$-module isomorphism $A:\mathbb{Z}_2\times\mathbb{Z}_2\to\mathbb{Z}_2\times\mathbb{Z}_2$ such that

$$\hat{T}=A^{-1}\circ T\times T\circ A.$$

Proof: Assume that such a $\mathbb{Z}_2$-module isomorphism A exists. Let $e_1=(1,0)$ and $e_2=(0,1)$. Then $Ae_1=(x_1,y_1)$ and $Ae_2=(x_2,y_2)$, where x_1,x_2,y_1, and y_2 are 2-adics and $(a,b)\in\mathbb{Z}_2\times\mathbb{Z}_2$. Then, for all $a,b\in\mathbb{Z}_2\times\mathbb{Z}_2$, $A\circ\hat{T}((a,b))=T\times T\circ A((a,b))$. Let $(a,b)\in(1,0)$.

$$\begin{aligned}T\times T(A((a,b))) &= A(\hat{T}((a,b)))\\ \Rightarrow T\times T\left(\begin{pmatrix}x_1 & x_2\\ y_1 & y_2\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}\right) &= A\left(\left(\frac{3a+1}{2},\frac{3b}{2}\right)\right)\\ \Rightarrow T\times T((ax_1+bx_2,ay_1+by_2)) &= \begin{pmatrix}x_1 & x_2\\ y_1 & y_2\end{pmatrix}\begin{pmatrix}\frac{3a+1}{2}\\ \frac{3b}{2}\end{pmatrix}\\ \Rightarrow T\times T((ax_1+bx_2,ay_1+by_2)) &= \left(\frac{3ax_1+3bx_2+x_1}{2},\frac{3ay_1+3by_2+y_1}{2}\right).\end{aligned}$$

Thus, we have

$$T\times T((ax_1+bx_2,ay_1+by_2))=\left(\frac{3ax_1+3bx_2+x_1}{2},\frac{3ay_1+3by_2+y_1}{2}\right).$$

In order to evaluate $T\times T((ax_1+bx_2,ay_1+by_2))$, we must determine the parities of ax_1+bx_2 and ay_1+by_2. Because b is even and a is odd, the parities are completely determined by, and equivalent to, the parities of x_1 and y_1. This yields the following four cases:

$$T\times T((ax_1+bx_2, ay_1+by_2)) = \begin{cases} \left(\frac{ax_1+bx_2}{2}, \frac{ay_1+by_2}{2}\right) & \text{if } x_1 \text{ even, } y_1 \text{ even,} \\ \left(\frac{3ax_1+3bx_2+1}{2}, \frac{ay_1+by_2}{2}\right) & \text{if } x_1 \text{ odd, } y_1 \text{ even,} \\ \left(\frac{ax_1+bx_2}{2}, \frac{3ay_1+3by_2+1}{2}\right) & \text{if } x_1 \text{ even, } y_1 \text{ odd,} \\ \left(\frac{3ax_1+3bx_2+1}{2}, \frac{3ay_1+3by_2+1}{2}\right) & \text{if } x_1 \text{ odd, } y_1 \text{ odd.} \end{cases}$$

From this it is easy to check that

$$T\times T((ax_1+bx_2, ay_1+by_2)) = \left(\frac{3ax_1+3bx_2+x_1}{2}, \frac{3ay_1+3by_2+y_1}{2}\right)$$

if and only if $x_1 = 1$ and $y_1 = 1$. Thus, A must be of the form $\left(\begin{smallmatrix}1 & x_2\\ 1 & y_2\end{smallmatrix}\right)$. Similarly, choosing $(a,b)\in(0,1)$ implies $x_2 = y_2 = 1$.

This means $A = \left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right)$, but $\left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right)$ is not invertible, which contradicts our assumption. □

Since $\hat{T}$ is conjugate to $\widetilde{T}$ via the $\mathbb{Z}_2$-module isomorphism B, we have

Corollary 1: $\widetilde{T}$ is not conjugate to $T\times T$ via a $\mathbb{Z}_2$-module isomorphism.

4. EXTENSION OF Q_k AND Q_∞ TO $\mathbb{Z}_2[i]$

One of our main reasons for extending to $\mathbb{Z}_2[i]$ was to add the tools associated with $\mathbb{Z}_2[i]$ to the current tools for studying the $3x+1$ problem. With this in mind, we redefine Q_k in terms of $\widetilde{T}$. Let $\widetilde{Q}_k:\mathbb{Z}_2[i]\to\mathbb{Z}_2[i]/2^k\mathbb{Z}_2[i]$ by $\widetilde{Q}_k(\alpha) = \widetilde{x}_0(\alpha), \widetilde{x}_1(\alpha), \widetilde{x}_2(\alpha), \ldots, \widetilde{x}_{k-1}(\alpha)$, where $\widetilde{x}_i(\alpha)\equiv \widetilde{T}^{(i)}(\alpha) \bmod 2$ for all $i\ge 0$ and $\widetilde{x}_i(\alpha)\in\{0,1,i,1+i\}$ be the *parity vector of length k for $\widetilde{T}$ of α*.

As with Q_k, $\widetilde{Q}_k$ completely describes the behavior of the first k iterates of α under $\widetilde{T}$.

We also define $\widetilde{Q}_\infty:\mathbb{Z}_2[i]\to\mathbb{Z}_2[i]$ in a similar manner and note that, as one would expect, $\widetilde{Q}_\infty$ completely describes the behavior of all iterates of α under $\widetilde{T}$.

$\widetilde{Q}_k$ and $\widetilde{Q}_\infty$ have properties similar to Q_k and Q_∞ as will be demonstrated in the following theorems which mirror analogous theorems for Q_k and Q_∞ found in [5].

Theorem 2: The function $\widetilde{Q}_k:\mathbb{Z}_2[i]\to\mathbb{Z}_2[i]/2^k\mathbb{Z}_2[i]$ is periodic with period 2^k.

In order to show that $\widetilde{Q}_k$ is periodic, we begin by showing

Lemma 1: $\widetilde{T}^k(\alpha+\omega 2^k)\equiv\widetilde{T}^k(\alpha)+\omega \bmod 2$, for any $\alpha,\omega\in\mathbb{Z}_2[i]$.

Proof: We proceed by induction on k. Let $\alpha,\omega\in\mathbb{Z}_2[i]$.

Base Case (k = 1). In this case,

$$\widetilde{T}(\alpha+\omega 2) = \begin{cases} \frac{\alpha}{2}+\omega\equiv\widetilde{T}(\alpha)+\omega & \bmod 2 & \text{if } \alpha\in[0], \\ \frac{3\alpha+1}{2}+3\omega\equiv\widetilde{T}(\alpha)+\omega & \bmod 2 & \text{if } \alpha\in[1], \\ \frac{3\alpha+i}{2}+3\omega\equiv\widetilde{T}(\alpha)+\omega & \bmod 2 & \text{if } \alpha\in[i], \\ \frac{3\alpha+1+i}{2}+3\omega\equiv\widetilde{T}(\alpha)+\omega & \bmod 2 & \text{if } \alpha\in[1+i]. \end{cases}$$

General Case: Assume $\widetilde{T}^{k-1}(\alpha+\omega 2^{k-1}) \equiv \widetilde{T}^{k-1}(\alpha)+\omega \mod 2$ for all n (inductive hypothesis).

Case 1: $\alpha \in [0]$. Then

$$\begin{aligned}
\widetilde{T}^{k}(\alpha+\omega 2^k) &= \widetilde{T}^{k-1}(\widetilde{T}(\alpha+\omega 2^k)) \\
&= \widetilde{T}^{k-1}\left(\tfrac{\alpha+\omega 2^k}{2}\right) && \text{(since } \alpha \in [0]) \\
&= \widetilde{T}^{k-1}\left(\tfrac{\alpha}{2}+\omega 2^{k-1}\right) \\
&\equiv \widetilde{T}^{k-1}\left(\tfrac{\alpha}{2}\right)+\omega \mod 2 && \text{(by ind. hyp.)} \\
&\equiv \widetilde{T}^{k}(\alpha)+\omega \mod 2 && \text{(since } \alpha \in [0]).
\end{aligned}$$

Case 2: $\alpha \in [1]$. Then

$$\begin{aligned}
\widetilde{T}^{k}(\alpha+\omega 2^k) &= \widetilde{T}^{k-1}(\widetilde{T}(\alpha+\omega 2^k)) \\
&= \widetilde{T}^{k-1}\left(\tfrac{3(\alpha+\omega 2^k)+1}{2}\right) && \text{(since } \alpha \in [1]) \\
&= \widetilde{T}^{k-1}\left(\tfrac{3\alpha+1}{2}+\omega 2^k+\omega 2^{k-1}\right) \\
&\equiv \widetilde{T}^{k-1}\left(\tfrac{3\alpha+1}{2}+\omega 2^k\right)+\omega \mod 2 && \text{(by ind. hyp.)} \\
&\equiv \widetilde{T}^{k-1}\left(\tfrac{3\alpha+1}{2}+\omega 2^{k-1}\right) \mod 2 && \text{(by ind. hyp.)} \\
&\equiv \widetilde{T}^{k-1}\left(\tfrac{3\alpha+1}{2}\right)+\omega \mod 2 && \text{(by ind. hyp.)} \\
&\equiv \widetilde{T}^{k}(\alpha)+\omega \mod 2 && \text{(since } \alpha \in [1]).
\end{aligned}$$

Case 3 ($\alpha \in [i]$) and Case 4 ($\alpha \in [1+i]$) are very similar to this case. Thus, $\widetilde{T}^{k}(\alpha+\omega 2^k) \equiv \widetilde{T}^{k}(\alpha)+\omega \mod 2$ for all k by induction on k. □

It follows easily that $\widetilde{x}_k$ is also periodic in the same sense.

Corollary 2: For every $\alpha, \omega \in \mathbb{Z}_2[i]$, $\widetilde{x}_j(\alpha+\omega 2^j) \equiv \widetilde{x}_j(\alpha)+\omega \mod 2$ for all $0 \le j \le \infty$.

From this we obtain Theorem 2.

Proof of Theorem 2: We proceed using induction on k.

Base Case (k = 1):

$$\begin{aligned}
\widetilde{Q}_1(\alpha+2\omega) &= \widetilde{x}_0(\alpha+2\omega) \\
&= \widetilde{x}_0(\alpha) && \text{(by Cor. 2)} \\
&= \widetilde{Q}_1(\alpha).
\end{aligned}$$

General Case: Assume $\widetilde{Q}_{k-1}(\alpha+\omega 2^{k-1}) = \widetilde{Q}_{k-1}(\alpha)$ (by inductive hypothesis). Then

$$\begin{aligned}
\widetilde{Q}_k(\alpha+\omega 2^k) &= \textstyle\sum_{j=0}^{k-1} \widetilde{x}_j(\alpha+\omega 2^k)2^j \\
&= \textstyle\sum_{j=0}^{k-2} \widetilde{x}_j(\alpha+\omega 2^k)2^j + \widetilde{x}_{k-1}(\alpha+\omega 2^k)2^{k-1} \\
&= \widetilde{Q}_{k-1}(\alpha+\omega 2^k) + \widetilde{x}_{k-1}(\alpha)2^{k-1} && \text{(by Cor. 2)} \\
&= \widetilde{Q}_{k-1}(\alpha) + \widetilde{x}_{k-1}(\alpha)2^{k-1} && \text{(by ind. hyp.)} \\
&= \textstyle\sum_{j=0}^{k-2} \widetilde{x}_j(\alpha)2^j + \widetilde{x}_{k-1}(\alpha)2^{k-1} \\
&= \textstyle\sum_{j=0}^{k-1} \widetilde{x}_j(\alpha)2^j = \widetilde{Q}_k(\alpha),
\end{aligned}$$

as required. □

Theorem 3: $\widetilde{Q}_\infty$ is an isometric homeomorphism.

Proof: We begin by showing that $\widetilde{Q}_\infty$ is one-to-one.

Let $\alpha, \beta \in \mathbb{Z}_2[i]$, $\alpha \neq \beta$. Then there exists $\omega \in \mathbb{Z}_2[i]$ such that $\alpha = \beta + \omega 2^k$, where $k = \min\{j : \alpha_j \neq \beta_j\}$, $\alpha = \alpha_0, \alpha_1, \ldots$, $\beta = \beta_0, \beta_1, \ldots$, and ω is not equivalent to 0 mod 2. By Corollary 2, $\widetilde{x}_k(\alpha) \equiv \widetilde{x}_k(\beta + \omega 2^k) \equiv \widetilde{x}_k(\beta) + \omega$ mod 2. Consequently, $\widetilde{x}_k(\alpha) - \widetilde{x}_k(\beta) \equiv \omega$ mod 2. Since ω is not equivalent to 0 mod 2, $\widetilde{x}_k(\alpha) \neq \widetilde{x}_k(\beta)$ and, therefore, by definition of $\widetilde{Q}_\infty$, $\widetilde{Q}_\infty(\alpha) \neq \widetilde{Q}_\infty(\beta)$. Thus, $\widetilde{Q}_\infty$ is one-to-one.

Next, we show that $\widetilde{Q}_\infty$ preserves the metric (and is therefore continuous).

Let $\alpha, \beta \in \mathbb{Z}_2[i]$, where $\alpha = \alpha_0, \alpha_1, \ldots, \beta = \beta_0, \beta_1, \ldots$. Choose k so that $D(\alpha, \beta) = 2^{-k}$. Then $\alpha \equiv \beta$ mod 2^k. By Theorem 2, $\widetilde{Q}_k(\alpha) = \widetilde{Q}_k(\beta)$, so $\widetilde{Q}_\infty(\alpha) \equiv \widetilde{Q}_\infty(\beta)$ mod 2^k. Thus, $D(\widetilde{Q}_\infty(\alpha), \widetilde{Q}_\infty(\beta)) \leq 2^{-k}$. However, because α is not equivalent to β mod 2^{k+1}, $\alpha = \beta + \omega 2^k$ for some $\omega \in \mathbb{Z}_2[i]$, where ω is not equivalent to 0 mod 2. Hence, $\widetilde{x}_k(\alpha) = \widetilde{x}_k(\beta + \omega 2^k) \equiv \widetilde{x}_k(\beta) + \omega$ mod 2 by Corollary 2. However, because ω is not equivalent to 0 mod 2, $\widetilde{x}_k(\alpha) \neq \widetilde{x}_k(\beta)$. It follows that $\widetilde{Q}_\infty(\alpha)$ is not equivalent to $\widetilde{Q}_\infty(\beta)$ mod 2^{k+1}. Therefore, $D(\widetilde{Q}_\infty(\alpha), \widetilde{Q}_\infty(\beta)) = 2^{-k}$ and so $\widetilde{Q}_\infty$ preserves the metric.

Finally, we show that $\widetilde{Q}_\infty$ is onto.

Let $\alpha = \alpha_0, \alpha_1, \ldots \in \mathbb{Z}_2[i]$, $\alpha'_k = \alpha_0, \ldots, \alpha_k, \overline{0} \in \mathbb{Z}_2[i]$, and let $\hat{\alpha}_k \in \mathbb{Z}_2[i]/2^k\mathbb{Z}_2[i]$ such that $\alpha \in \hat{\alpha}_k$. We first note that $\widetilde{Q}_k$ is onto, as can be seen by induction on k using Corollary 2. There exists a β'_k such that $\widetilde{Q}_k(\beta'_k) = \hat{\alpha}_k$. Let $\beta'_k = \beta_0, \ldots, \beta_k, \overline{0}$. We can see that $\widetilde{Q}_\infty(\beta'_k) = \hat{\alpha}'_k$ mod 2^k. Thus, $\lim_{k\to\infty} D(\widetilde{Q}_\infty(\beta), \alpha'_k)) = 0$. Consequently, $\lim_{k\to\infty} \widetilde{Q}_\infty(\beta'_k) = \lim_{k\to\infty} \alpha'_k = \alpha$. Now, since $\widetilde{Q}_\infty$ is continuous, $\lim_{k\to\infty} \widetilde{Q}_\infty(\beta'_k) = \widetilde{Q}_\infty(\lim_{k\to\infty} \beta'_k) = \alpha$. Therefore, all that remains is to show that $\lim_{k\to\infty} \beta'_k$ exists as a Gaussian 2-adic. Since the sequence $\{\widetilde{Q}_\infty(\beta'_k)\}$ converges to α, it is Cauchy. Because $\widetilde{Q}_\infty$ preserves the metric, the sequence $\{\beta'_k\}$ in $\mathbb{Z}_2[i]$ is also a Cauchy sequence. Now, $\mathbb{Z}_2[i]$ is a compact metric space by the Tychonoff theorem, so every Cauchy sequence in $\mathbb{Z}_2[i]$ has a limit in $\mathbb{Z}_2[i]$. Thus, the sequence $\{\beta'_k\}$ converges to some $\beta \in \mathbb{Z}_2[i]$ and $\widetilde{Q}_\infty(\beta) = \alpha$. Therefore, $\widetilde{Q}_\infty$ is onto.

$\widetilde{Q}_\infty^{-1}$ is continuous because $\widetilde{Q}_\infty$ is an isometry; thus, $\widetilde{Q}_\infty$ is an isometric homeomorphism. □

Now that we have shown $\widetilde{Q}_\infty$ is a continuous bijection, we shall see just how powerful a tool it is in our exploration of the dynamics of $\widetilde{T}$.

5. CHAOS AND THE 3x + 1 PROBLEM

A map $F:(X,d) \to (X,d)$ is defined to be *transitive* if $\forall x, y \in X$, $\forall \varepsilon > 0$, $\exists z \in X$ such that $d(x,z) < \varepsilon$ and $d(y, F^{(k)}(z)) < \varepsilon$ for some $k \geq 0$ [4]. Devaney [4] defines a *chaotic map* to be a transitive map with dense periodic points. Chaoticity in this sense is preserved by topological conjugacy, so we can show that a function is chaotic if it is topologically conjugate to a known chaotic map. Such a map is the shift map on the sequence space, $\sigma : \Sigma \to \Sigma$, where

$$\Sigma = \{s_0, s_1, s_2, \ldots | s_j \in \{0, 1\}\} \quad \text{and} \quad \sigma(s_0, s_1, s_2, \ldots) = s_1, s_2, s_3, \ldots.$$

Q_∞ provides a conjugacy between T and σ. Thus, we have shown

Theorem 4: $T:\mathbb{Z}_2 \to \mathbb{Z}_2$ is chaotic.

It should be noted, however, that the restriction of T to $\mathbb{Z}^+$, an invariant subdomain of $\mathbb{Z}_2$, is not chaotic, according to the $3x+1$ conjecture.

Since the product of chaotic maps is itself a chaotic map, we have

Corollary 3: $T \times T$ is chaotic.

Certainly if T is chaotic, any reasonable extension of T should also be chaotic. We show that $\widetilde{T}$ is chaotic by showing that it is conjugate to some known chaotic system. With this in mind, we define $\sigma_4:(\Sigma_4, d_\delta) \to (\Sigma_4, d_\delta)$ and show that σ_4 and $\sigma \times \sigma$ are conjugate via a homeomorphism, F.

Let $\sigma_4:(\Sigma_4, d_\delta) \to (\Sigma_4, d_\delta)$ be the shift map on the sequence space with four elements $\{0, 1, i, i+1\}$, where

$$d_\delta((s_0, s_1, \dots), (t_0, t_1, \dots)) = \sum_{k=0}^{\infty} \frac{\delta_k(s,t)}{4^k}$$

and

$$\delta_k = \begin{cases} 0 & \text{if } s_k = t_k, \\ 1 & \text{otherwise.} \end{cases}$$

It can easily be shown that $\sigma_4:(\Sigma_4, d_\delta) \to (\Sigma_4, d_\delta)$ is chaotic.

Lemma 2: The function $F:(\Sigma_4, d_\delta) \to \Sigma \times \Sigma$ by

$$F(s) = ((a_1(s), a_2(s), a_3(s), \dots), (b_1(s), b_2(s), b_3(s), \dots)),$$

where each

$$a_i(s) = \begin{cases} 0 & \text{if } s_i = 0 \text{ or } i, \\ 1 & \text{if } s_i = 1 \text{ or } 1+i, \end{cases}$$

and each

$$b_i(s) = \begin{cases} 0 & \text{if } s_i = 0 \text{ or } 1, \\ 1 & \text{if } s_i = i \text{ or } 1+i, \end{cases}$$

is a homeomorphism.

Proof: It is clear that F is a bijection. We now show that F is continuous. Let $\varepsilon > 0$, $\delta = 4^{-k}$, where k is chosen to make $2^{-k} < \varepsilon$, $s = s_0, s_1, \dots \in \Sigma_4$, $t = t_0, t_1, \dots \in \Sigma_4$. If $d_\delta(s,t) < \delta$, then $s_j = t_j$ for all $0 \le j \le k$. Consequently, if we consider that $F(s) = (x, y)$ and $F(t) = (z, w)$ for some $(x,y), (z,w) \in \Sigma \times \Sigma$, where $x = x_0, x_1, \dots \in \Sigma$, $y = y_0, y_1, \dots \in \Sigma$, $z = z_0, z_1, \dots \in \Sigma$, and $w = w_0, w_1, \dots \in \Sigma$, then $x_j = z_j$ and $y_j = w_j$ for all $0 \le j \le k$ by definition of d_δ. Thus, by definition of F, $d_x(F(s), F(t)) \le 2^{-k} < \varepsilon$, where d_x is the product metric on Σ. So F is continuous.

By letting $\varepsilon > 0$ and choosing $\delta = 2^{-k}$, where k is such that $4^{-k} < \varepsilon$, we can apply a similar argument to show that F^{-1} is continuous. Therefore, F is a homeomorphism. □

It easily follows that

Corollary 4: σ_4 and $\sigma \times \sigma$ are conjugate via F.

Since $\widetilde{T}$ is conjugate to σ_4 via $\widetilde{Q}_\infty$, we have

Theorem 5: $\widetilde{T}$ is chaotic.

It turns out that, in proving the chaoticity of T, $\widetilde{T}$, and $T \times T$, we have defined some very useful conjugacies, as we shall see in the next section.

6. RELATIONSHIP BETWEEN $\widetilde{T}$ AND $T \times T$

Though $\widetilde{T}$ and $T \times T$ are not conjugate via a $\mathbb{Z}_2$-module isomorphism, they are topologically conjugate. Since topological conjugacy is transitive, $\widetilde{T} \cong \sigma_4 \cong \sigma \times \sigma \cong T \times T$, where $\cong$ denotes topological conjugacy and thus

Theorem 6: $\widetilde{T}$ and $T \times T$ are topologically conjugate (via $(Q_\infty \times Q_\infty)^{-1} \circ F \circ \widetilde{Q}_\infty$).

These theorems allow us to work in the system of our choice and then convert the results to any other system using the homeomorphisms defined above.

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