# A CLOSED FORM OF THE (2, F) GENERALIZATIONS OF THE FIBONACCI SEQUENCE

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## 1. INTRODUCTION

In this paper we consider the generalized (2, $F$) sequences. They are introduced in [1] and [2], and some of their properties are studied in [1], [2], [5], [7], [8], and [9]. The generalized (2, $F$) sequences $\{x_i\}_{i=0}^{\infty}$ and $\{y_i\}_{i=0}^{\infty}$ are defined by their first two elements and two linear equalities:

$$x_0 = a,\ x_1 = b,\ y_0 = c,\ y_1 = d,$$
$$x_{n+2} = \alpha x_{n+1} + \beta y_n,\ y_{n+2} = \gamma y_{n+1} + \delta x_n,$$

for $n \geq 0$. In [1] the following open problem is given: Find a closed form of $x_n$ and $y_n$ for arbitrary $n$, i.e., represent them as functions of $n$, $a$, $b$, $c$, $d$, $\alpha, \beta, \gamma$, and $\delta$. In [5] such functions are obtained. They have one of the following five forms:

$$\begin{aligned}
x_n &= C_1\rho_1^n + C_2\rho_2^n + C_3\rho_3^n + C_4\rho_4^n, & y_n &= C_5\rho_1^n + C_6\rho_2^n + C_7\rho_3^n + C_8\rho_4^n, && \text{or}\\
x_n &= C_1\rho_1^n + C_2\rho_2^n + (C_3 + nC_4)\rho_3^n, & y_n &= C_5\rho_1^n + C_6\rho_2^n + (C_7 + nC_8)\rho_3^n, && \text{or}\\
x_n &= C_1\rho_1^n + (C_2 + C_3 n + C_4 n^2)\rho_2^n, & y_n &= C_5\rho_1^n + (C_6 + C_7 n + C_8 n^2)\rho_2^n, && \text{or}\\
x_n &= (C_1 + C_2 n + C_3 n^2 + C_4 n^3)\rho_1^n, & y_n &= (C_5 + C_6 n + C_7 n^2 + C_8 n^3)\rho_1^n, && \text{or}\\
x_n &= (C_1 + C_2 n)\rho_1^n + (C_3 + C_4 n)\rho_2^n, & y_n &= (C_5 + C_6 n)\rho_1^n + (C_7 + C_8 n)\rho_2^n,
\end{aligned}$$

where $\rho_1, \rho_2, \rho_3$, and $\rho_4$ are the roots (complex in the general case) of the equation

$$\rho^4 - (\alpha + \gamma)\rho^2 + \alpha\gamma\rho - \beta\delta = 0$$

(the above five cases correspond to four simple roots, two simple roots and one double root, ..., two double roots, respectively) and $C_i$, $1 \leq i \leq 8$ are (complex) constants depending on $a$, $b$, $c$, $d$, and $\rho_i$, $1 \leq i \leq 4$.

We shall give an alternative closed form for $x_n$ and $y_n$. Our approach is fully combinatorial (it is based on an enumeration of weighted paths in an infinite graph) whereas the Georgieu-Atanassov method is from linear algebra (it uses Jordan's factorization form of some matrix). More concretely, we shall prove the following.

***Theorem 1 (Main result):*** The equalities

$$x_n = a \sum_{4p+q+r=n-4} \binom{p+q}{q}\binom{p+r}{r} \alpha^q \beta^{p+1} \gamma^r \delta^{p+1} + b \sum_{4p+q+r=n-1} \binom{p+q}{q}\binom{p+r-1}{r} \alpha^q \beta^p \gamma^r \delta^p$$
$$+ c \sum_{4p+q+r=n-2} \binom{p+q}{q}\binom{p+r-1}{r} \alpha^q \beta^{p+1} \gamma^r \delta^p + d \sum_{4p+q+r=n-3} \binom{p+q}{q}\binom{p+r}{r} \alpha^q \beta^{p+1} \gamma^r \delta^p$$

and

$$y_n = a \sum_{4p+q+r=n-2} \binom{p+q-1}{q}\binom{p+r}{r} \alpha^q \beta^p \gamma^r \delta^{p+1} + b \sum_{4p+q+r=n-3} \binom{p+q}{q}\binom{p+r}{r} \alpha^q \beta^p \gamma^r \delta^{p+1}$$

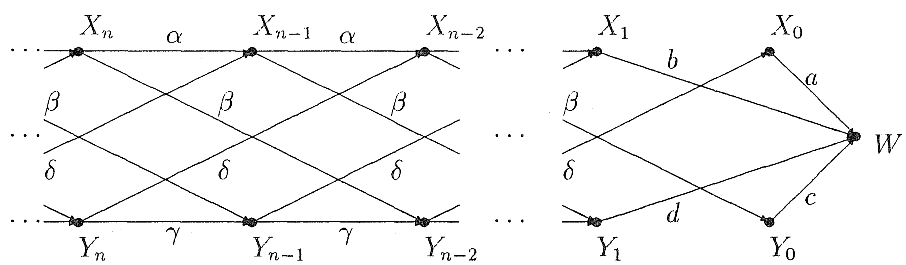
$$+ c \sum_{4p+q+r=n-4} \binom{p+q-1}{q}\binom{p+r}{r} \alpha^q \beta^{p+1} \gamma^r \delta^{p+1} + d \sum_{4p+q+r=n-1} \binom{p+q-1}{q}\binom{p+r}{r} \alpha^q \beta^p \gamma^r \delta^p$$

hold for every $n \geq 2$, where all sums are taken for nonnegative integer values of $p$, $q$, and $r$.

## 2. PROOF OF THE MAIN RESULT

Our basic construction is an infinite directed graph $G = (V, E)$ with weighted edges:

The set of vertices is $V = \{W\} \cup \{X_i \mid i \in Z_{\geq 0}\} \cup \{Y_i \mid i \in Z_{\geq 0}\}$ (here $Z_{\geq 0}$ denotes the set of nonnegative integers). The set of edges is $E = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_0$, where $E_1 = \{(X_i, X_{i-1}) \mid i \geq 2\}$, all edges from $E_1$ have weight $\alpha$ and we shall call them edges of type $A$. Analogously, the set of edges of type $B$ with weight $\beta$ is $E_2 = \{(X_i, Y_{i-2}) \mid i \geq 2\}$, the set of edges of type $C$ with weight $\gamma$ is $E_3 = \{(Y_i, Y_{i-1}) \mid i \geq 2\}$, and the set of edges of type $D$ with weight $\delta$ is $E_4 = \{(Y_i, X_{i-2}) \mid i \geq 2\}$. The last set $E_0$ consists of the following four edges: $(X_1, W)$ with weight $a$, $(X_0, W)$ with weight $b$, $(Y_1, W)$ with weight $c$, and $(Y_0, W)$ with weight $d$. A graphical representation of $G$ is given in the figure below.




We define the weight of a path in $G$ as the product of weights of its edges. For two arbitrary vertices $v_1, v_2 \in V$, $v_1 \neq v_2$, we define the function $\omega(v_1, v_2)$ as the sum of the weights of all paths from $v_1$ to $v_2$ in $G$; for $v_1 = v_2$, we set $\omega(v_1, v_2) = 1$. The following lemma shows the connection between function $\omega$ and sequences $\{x_i\}_{i=0}^{\infty}$, $\{y_i\}_{i=0}^{\infty}$.

***Lemma 1:*** $\omega(X_i, W) = x_i$ and $\omega(Y_i, W) = y_i$ hold for every $i \in Z_{\geq 0}$.

***Proof:*** The proof is straightforward by induction on $i$. For $i \in \{0, 1\}$, we have $\omega(X_0, W) = a$, $\omega(x_1, W) = b$, $\omega(Y_0, W) = c$, and $\omega(Y_1, W) = d$. For $i \geq 2$, we observe that every path from $X_i$ to $W$ starts with the edge $(X_i, X_{i-1})$ or with the edge $(X_i, Y_{i-2})$. Thus, $\omega(X_i, W) = \alpha\omega(X_{i-1}, W) + \beta\omega(Y_{i-2}, W) = \alpha x_{i-1} + \beta y_{i-2} = x_i$. The proof for $\omega(Y_i, W)$ is similar. □

We shall compute some values of the function $\omega$ that we shall use further.

***Lemma 2:*** The following equalities hold for every $i, j \in \mathbf{Z}$, $i \geq j \geq 1$ (all sums are taken for nonnegative integer values of $p$, $q$, and $r$):

*1.* $\omega(X_i, X_j) = \sum_{4p+q+r=i-j} \binom{p+q}{q}\binom{p+r-1}{r} \alpha^q \beta^p \gamma^r \delta^p,$

2. $\omega(Y_i, Y_j) = \sum_{4p+q+r=i-j} \binom{p+q-1}{q}\binom{p+r}{r}\alpha^q\beta^p\gamma^r\delta^p,$

3. $\omega(X_i, Y_j) = \sum_{4p+q+r=i-j-2} \binom{p+q}{q}\binom{p+r}{r}\alpha^q\beta^{p+1}\gamma^r\delta^p,$

4. $\omega(Y_i, X_j) = \sum_{4p+q+r=i-j-2} \binom{p+q}{q}\binom{p+r}{r}\alpha^q\beta^p\gamma^r\delta^{p+1}.$

***Proof:*** We shall prove case 1 only; the proofs of 2, 3, and 4 are similar.

Let us consider the structure of an arbitrary path from $X_i$ to $X_j$. Edges of type $B$ and $D$ alternate, starting with an edge of type $B$ and ending with an edge of type $D$. It is clear also that there are edges of type $C$ only between neighboring pairs ($B$, $D$) and there are edges of type $A$ only between neighboring pairs ($D$, $B$) at the beginning and at the end. Therefore, the considered path has the form

$$\underbrace{A\ldots A}_{q_1}\,B\,\underbrace{C\ldots C}_{r_1}\,D\,\underbrace{A\ldots A}_{q_2}\,B\,\underbrace{C\ldots C}_{r_2}\,D\ldots\underbrace{A\ldots A}_{q_p}\,B\,\underbrace{C\ldots C}_{r_p}\,D\,\underbrace{A\ldots A}_{q_{p+1}},$$

where the number of edges of types $B$ and $D$ is $p$, the number of edges of type $A$ is $q=\sum_{k=1}^{p+1}q_k$, and the number of edges of type $C$ is $r=\sum_{k=1}^{p}r_k$. It is known that the number of all nonnegative ordered $p+1$-tuples with sum $q$ is $\binom{p+q}{q}$ and the number of all nonnegative ordered $p$-tuples with sum $r$ is $\binom{p+r-1}{r}$. Since the tuples $(q_1, q_2, \ldots, q_{p+1})$ and $(r_1, r_2, \ldots, r_p)$ are independent, we obtain that the total number of paths from $X_i$ to $X_j$ with $q$ edges of type $A$, $p$ edges of type $B$, $r$ edges of type $C$, and $p$ edges of type $D$ is $\binom{p+q}{q}\binom{p+r-1}{r}$. Their weight is $\alpha^q\beta^p\gamma^r\delta^p$. Thus, we need all admissible values of $p$, $q$, and $r$ to compute $\omega(X_i, X_j)$. Since the difference between indices of the vertices adjacent to the edge of type $B$ or $D$ is 2 and the difference for the edges of type $A$ or $C$ is 1, we have that $i-j=4p+q+r$. That is why we obtain

$$\omega(X_i, X_j) = \sum_{4p+q+r=i-j} \binom{p+q}{q}\binom{p+r-1}{r}\alpha^q\beta^p\gamma^r\delta^p,$$

where the sum is taken for nonnegative integer values of $p$, $q$, and $r$. □

Now we are able to prove our main result (Theorem 1).

***Proof of the Main Result:*** Let us observe that the last edge of an arbitrary path from $X_n$ to $W$ is $(X_0, W)$ or $(X_1, W)$ or $(Y_0, W)$ or $(Y_1, W)$. Thus,

$$x_n = \omega(X_n, W) = a\omega(X_n, X_0) + b\omega(X_n, X_1) + c\omega(X_n, Y_0) + d\omega(X_n, Y_1).$$

Let us observe also that every path from $X_n$ to $X_0$ ends with edge $(Y_2, X_0)$ and every path from $X_n$ to $Y_0$ ends with edge $(X_2, Y_0)$. That is why

$$x_n = a\delta\omega(X_n, Y_2) + b\omega(X_n, X_1) + c\beta\omega(X_n, X_2) + d\omega(X_n, Y_1).$$

The proof for $y_n$ is similar. □

Finally, we mention that some other problems from [1]-[11] can also be solved using the described method.

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