ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to stan@mathpro.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

The Fibonacci polynomials $F_n(x)$ and the Lucas polynomials $L_n(x)$ satisfy

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x),\quad F_0(x) = 0,\quad F_1(x) = 1;$$
$$L_{n+2}(x) = xL_{n+1}(x) + L_n(x),\quad L_0(x) = 2,\quad L_1(x) = x.$$

Also,

$$F_n(x) = \frac{\alpha(x)^n - \beta(x)^n}{\alpha(x) - \beta(x)} \quad \text{and} \quad L_n(x) = \alpha(x)^n + \beta(x)^n,$$

where $\alpha(x) = (x + \sqrt{x^2+4})/2$ and $\beta(x) = (x - \sqrt{x^2+4})/2$.

PROBLEMS PROPOSED IN THIS ISSUE

B-860 ***Proposed by Herta T. Freitag, Roanoke, VA***

Let k be a positive integer. The sequence $\langle A_n \rangle$ is defined by the recurrence $A_{n+2} = 2kA_{n+1} - A_n$ for $n \geq 0$ with initial conditions $A_0 = 0$ and $A_1 = 1$. Prove that $(k^2-1)A_n^2 + 1$ is a perfect square for all $n \geq 0$.

B-861 ***Proposed by the editor***

The sequence $w_0, w_1, w_2, w_3, w_4, \ldots$ satisfies the recurrence $w_n = Pw_{n-1} - Qw_{n-2}$ for $n > 1$. If every term of the sequence is an integer, must P and Q both be integers?

B-862 ***Proposed by Charles K. Cook, University of South Carolina, Sumter, SC***

Find a Fibonacci number and a Lucas number whose sum is 114,628 and whose least common multiple is 567,451,586.

B-863 ***Proposed by Stanley Rabinowitz, Westford, MA***

Let

$$A=\begin{pmatrix}-9 & 1\\ -89 & 10\end{pmatrix},\quad B=\begin{pmatrix}-10 & 1\\ -109 & 11\end{pmatrix},\quad C=\begin{pmatrix}-7 & 5\\ -11 & 8\end{pmatrix},\quad \text{and}\quad D=\begin{pmatrix}-4 & 19\\ -1 & 5\end{pmatrix},$$

and let n be a positive integer. Simplify $30A^n-24B^n-5C^n+D^n$.

B-864 ***Proposed by Stanley Rabinowitz, Westford, MA***

The sequence $\langle Q_n\rangle$ is defined by $Q_n=2Q_{n-1}+Q_{n-2}$ for $n>1$ with initial conditions $Q_0=2$ and $Q_1=2$.

(a) Show that $Q_{7n}\equiv L_n \pmod{159}$ for all n.
(b) Find an integer $m>1$ such that $Q_{11n}\equiv L_n \pmod{m}$ for all n.
(c) Find an integer a such that $Q_{an}\equiv L_n \pmod{31}$ for all n.
(d) Show that there is no integer a such that $Q_{an}\equiv L_n \pmod{7}$ for all n.
(e) Extra credit: Find an integer $m>1$ such that $Q_{19n}\equiv L_n \pmod{m}$ for all n.

B-865 ***Proposed by Alexandru Lupas, University Lucian Blaga, Sibiu, Romania***

Let $f(x)=(x^2+4)^{n-1/2}$ where n is a positive integer. Let

$$g(x)=\frac{d^n f(x)}{dx^n}.$$

Express $g(1)$ in terms of Fibonacci and/or Lucas numbers.

Note: The Elementary Problems Column is in need of more **easy**, yet elegant and non-routine problems.

SOLUTIONS

See "Basic Formulas" at the beginning of this column for notation about Fibonacci and Lucas polynomials.

It Repeats!

B-843 ***Proposed by R. Horace McNutt, Montreal, Canada***
(Vol. 36, no. 1, February 1998)

Find the last three digits of $L_{1998}(114)$.

Solution by L. A. G. Dresel, Reading, England

We note that when $x=114$, $x^2+4=13000$. From the recurrence for $L_n(x)$, we have $L_0(x)=2$, $L_1(x)=x$, $L_2(x)=x^2+2\equiv -2 \pmod{x^2+4}$ so that $L_3(x)\equiv -x$, $L_4(x)\equiv -x^2-2\equiv 2$, and $L_5(x)\equiv x \pmod{x^2+4}$.

Hence, modulo 13000, the sequence $L_n(114)$ is periodic with period 4, so that

$$L_{1998}(114)\equiv L_2(114)\equiv -2\equiv 12998 \pmod{13000}.$$

Therefore, the last three digits of $L_{1998}(114)$ are 998.

Solutions also received by Brian D. Beasley, Paul S. Bruckman, Mario DeNobili, Aloysius Dorp, Russell Jay Hendel, Harris Kwong, H.-J. Seiffert, Indulis Strazdins, and the proposer. One incorrect solution was received.

A Polynomial Identity

B-844 *Proposed by Mario DeNobili, Vaduz, Lichtenstein (Vol. 36, no. 1, February 1998)*

If $a+b$ is even and $a>b$, show that

$$[F_a(x)+F_b(x)][F_a(x)-F_b(x)] = F_{a+b}(x)F_{a-b}(x).$$

Solution 1 by Harris Kwong, SUNY College at Fredonia, Fredonia, NY

Since $a+b$ is even, a and b have the same parity, and $a-b$ is also even. For brevity, we shall write $\alpha(x)$, $\beta(x)$, $F_a(x)$, and $F_b(x)$ as α, β, F_a, and F_b, respectively. It follows from $\alpha\beta=-1$ that

$$\begin{aligned}(\alpha-\beta)^2(F_a^2-F_b^2) &= (\alpha^a-\beta^a)^2-(\alpha^b-\beta^b)^2\\ &= \alpha^{2a}-2(\alpha\beta)^a+\beta^{2a}-[\alpha^{2b}-2(\alpha\beta)^b+\beta^{2b}]\\ &= \alpha^{2a}+\beta^{2a}-\alpha^{2b}-\beta^{2b}.\end{aligned}$$

On the other hand,

$$\begin{aligned}(\alpha-\beta)^2F_{a+b}F_{a-b} &= (\alpha^{a+b}-\beta^{a+b})(\alpha^{a-b}-\beta^{a-b})\\ &= \alpha^{2a}-(\alpha\beta)^{a-b}(\beta^{2b}+\alpha^{2b})+\beta^{2a}\\ &= \alpha^{2a}+\beta^{2a}-\alpha^{2b}-\beta^{2b}.\end{aligned}$$

Therefore, $F_{a+b}F_{a-b}=F_a^2-F_b^2=[F_a+F_b][F_a-F_b]$.

Solution 2 by H.-J. Seiffert, Berlin, Germany

It is known ([1], p. 12, formula 3.26) that

$$F_{n+m}^2(x)-F_{n-m}^2(x)=F_{2n}(x)F_{2m}(x)$$

for all integers m and n. If $a+b$ is even, then $a-b$ is also even. With $n=(a+b)/2$ and $m=(a-b)/2$, the above identity gives the desired one.

Reference

1. A. F. Horadam & Bro. J. M. Mahon. "Pell and Pell-Lucas Polynomials." *The Fibonacci Quarterly* **23.1** (1985):7-20.

Comment by the editor: No reader sent in any generalizations related to Lucas polynomials. If $\langle v_n\rangle$ is the generalized Lucas sequence defined by the recurrence $v_n=Pv_{n-1}-Qv_{n-2}$ with initial conditions $v_0=2$ and $v_1=P$, then one can investigate the expression

$$(v_a+v_b)(v_a-v_b)-v_{a+b}v_{a-b}.$$

Applying Algorithm `LucasSimplify` (from [1]), shows that this expression simplifies to

$$Q^{-b}(4Q^{a+b}-Q^av_b^2-Q^bv_b^2).$$

Thus, if $a+b$ is odd, $a>b$, and $Q=-1$, then we have

$$(v_a+v_b)(v_a-v_b)-v_{a+b}v_{a-b}=4(-1)^a.$$

In particular, for the Lucas polynomials, we have $P=x$ and $Q=-1$. This shows that if $a+b$ is odd with $a>b$, then

$$[L_a(x)+L_b(x)][L_a(x)-L_b(x)]=L_{a+b}(x)L_{a-b}(x)+4(-1)^a.$$

Reference

1. Stanley Rabinowitz. "Algorithmic Manipulation of Fibonacci Identities." In *Applications of Fibonacci Numbers* **6**:389-408. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1996.

Solutions also received by Brian D. Beasley, Paul S. Bruckman, Leonard A. G. Dresel, Steve Edwards, Russell Euler & Jawad Sadek, Russell Jay Hendel, Harris Kwong, Gene Ward Smith, Lawrence Somer, Indulis Strazdins, and the proposer.

Curious Commuting Composition

B-845 ***Proposed by Gene Ward Smith, Brunswick, ME (Vol. 36, no. 1, February 1998)***

Show that, if m and n are odd positive integers, then $L_n(L_m(x)) = L_m(L_n(x))$.

Solution 1 by Harris Kwong, SUNY College at Fredonia, Fredonia, NY

Given x, choose $\theta = \theta(x)$ such that $x = 2i\sin\theta$. Then $\sqrt{x^2+4} = 2\cos\theta$, hence

$$\alpha(x) = \cos\theta + i\sin\theta \quad\text{and}\quad \beta(x) = -(\cos\theta - i\sin\theta).$$

Consequently, for any odd positive integer n, we have

$$L_n(x) = (\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta) = 2i\sin n\theta.$$

In other words, $\theta(L_n(x)) = n\theta(x)$. It follows immediately that if m and n are odd positive integers, then $L_m(L_n(x)) = 2i\sin mn\theta(x) = L_n(L_m(x))$.

Solution 2 by Indulis Strazdins, Riga, Latvia

Put $y = L_m(x)$ in the basic equality

$$L_n(y) = yL_{n-1}(y) + L_{n-2}(y)$$

to get

$$L_n(L_m(x)) = L_m(x)L_{n-1}(L_m(x)) + L_{n-2}(L_m(x)).$$

It is easily proved by induction that

$$L_{nm}(x) = L_m(x)L_{(n-1)m}(x) + (-1)^{m-1}L_{(n-2)m}(x)$$

so, if m is odd, we have

$$L_n(L_m(x)) = L_{nm}(x).$$

From this it follows that if m and n are both odd, then

$$L_n(L_m(x)) = L_m(L_n(x)) = L_{nm}(x).$$

Comment by L. A. G. Dresel, Reading, England: Di Porto and Filipponi (see [1], p. 221) have proven the following Lemma:

If m and n are integers with m odd, then $L_n(L_m(x)) = L_{nm}(x)$.

If m and n are both odd, then the desired result follows.

Reference

1. A. Di Porto & P. Filipponi. "A Probabilistic Primality Test Based on the Properties of Certain Generalized Lucas Numbers." *Lecture Notes in Computer Science* **330** (1988):211-23.

No reader submitted any related results for compositions of Fibonacci polynomials.

Solutions also received by Paul S. Bruckman, Leonard A. G. Dresel, Russell Euler & Jawad Sadek, R. Horace McNutt, H.-J. Seiffert, and the proposer.

Integer Sum

B-846 ***Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy (Vol. 36, no. 1, February 1998)***

Show that

$$\sum_{n=1}^{5} \frac{F_n(40k+1)}{n!}$$

is an integer for all integral k. Generalize.

Solution by Gene Ward Smith, Brunswick, ME

The sum in question,

$$\sum_{n=1}^{5} \frac{F_n(40k+1)}{n!},$$

is a polynomial of degree 4, namely,

$$(64000k^4 + 14400k^3 + 1760k^2 + 135k + 6)/3.$$

Writing this in terms of binomial coefficients gives us

$$2 + 26765\binom{k}{1} + 328640\binom{k}{2} + 796800\binom{k}{3} + 512000\binom{k}{4};$$

which has integer coefficients. The polynomial therefore is integer-valued for integer values of k. Alternatively, we may factor

$$64000k^4 + 14400k^3 + 1760k^2 + 135k + 6,$$

modulo 3, and obtain $k^2(k+1)(k+2)$, from which it follows that 3 is a divisor for any integer k, so that the initial polynomial is integer-valued.

We may generalize in various ways, most obviously by considering instead

$$\sum_{n=1}^{r} \frac{F_n(ak+b)}{n!}$$

for various values of a, b, and r. In this way we may, for instance, similarly prove that

$$\sum_{n=1}^{5} \frac{F_n(40k+9)}{n!}$$

is integral.

Bruckman and Seiffert showed that $\sum_{n=1}^{5} \frac{F_n(x)}{n!}$ is an integer if and only if $x \equiv 1$ or 9 (mod 40). Bruckman showed that $\sum_{i=1}^{n} \frac{F_i(x)}{i!}$ is never an integer if $n = 3$ or 4. The proposed stated that if k is a positive odd integer, then $\sum_{n=1}^{5} \frac{F_n^k(x)}{n!}$ is an integer if and only if $x \equiv 1$ or 9 (mod 40); but he did not include a proof.

Solutions also received by Paul S. Bruckman and H.-J. Seiffert.

Polynomial GCD

B-847 ***Proposed by Gene Ward Smith, Brunswick, ME***
(Vol. 36, no. 1, February 1998)

Find the greatest common polynomial divisor of $F_{n+4k}(x)+F_n(x)$ and $F_{n+4k-1}(x)+F_{n-1}(x)$.

Solution by Paul S. Bruckman, Highland, IL

Let $d=d(x)=\sqrt{x^2+4}$. For brevity, write $\alpha(x)$ as α and $\beta(x)$ as β. Note that $\alpha\beta=-1$. Then

$$\begin{aligned}F_{n+4k}(x)+F_n(x)&=\frac{\alpha^{n+4k}-\beta^{n+4k}}{d}+\frac{\alpha^n-\beta^n}{d}\\&=\alpha^n\frac{\alpha^{4k}+1}{d}-\beta^n\frac{\beta^{4k}+1}{d}\\&=\alpha^{n+2k}\frac{\alpha^{2k}+\beta^{2k}}{d}-\beta^{n+2k}\frac{\alpha^{2k}+\beta^{2k}}{d}\\&=\frac{\alpha^{n+2k}-\beta^{n+2k}}{d}\cdot(\alpha^{2k}+\beta^{2k})\\&=F_{n+2k}(x)L_{2k}(x).\end{aligned}$$

Replacing n by $n-1$, we find that

$$F_{n-1+4k}(x)+F_{n-1}(x)=F_{n-1+2k}(x)L_{2k}(x).$$

If g is the desired gcd of the two given expressions, then $g=L_{2k}(x)\cdot\gcd(F_{n+2k}(x),F_{n-1+2k}(x))$.

Given any integer u, if d is the gcd of $F_u(x)$ and $F_{u-1}(x)$, then by the recurrence relation, d is a common divisor of $F_1(x)=1$ and $F_2(x)=x$. Thus, $d=\gcd(F_{n+2k}(x),F_{n-1+2k}(x))=1$ and $g=L_{2k}(x)$.

Solutions also received by Leonard A. G. Dresel, H.-J. Seiffert, Indulis Strazdins, and the proposer.

Addenda. We wish to belatedly acknowledge solutions from the following solvers:

Brian Beasley—B-842
Glenn A. Bookhout—B-784
Andrej Dujella—B-772 through B-777
Steve Edwards—B-837, B-840, B-842
Russell Euler—B-788
Herta Freitag—B-791, B-793
Hans Kappus—B-784 through B-786
Carl Libis—B-784, B-785
Graham Lord—B-784, B-785

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