ON USING PATTERNS IN BETA-EXPANSIONS TO STUDY FIBONACCI-LUCAS PRODUCTS

Evelyn L. Hart
Dept. of Math., Colgate University, 13 Oak Drive, Hamilton, NY 13346-1398
(*Submitted January 1997*)

1. INTRODUCTION

The Zeckendorf decomposition of a natural number n is the unique expression of n as a sum of Fibonacci numbers with nonconsecutive indices and with each index greater than 1, where $F_0 = 0$, $F_1 = 1$, and $F_{i+2} = F_i + F_{i+1}$ form the Fibonacci numbers for $i \geq 0$ (see [13] and [17], or see [16, pp. 108-09]). The Zeckendorf decompositions of products of the forms kF_m and kL_m with $k, m \in \mathbb{N}$ (where $L_m = F_{m-1} + F_{m+1}$ is the m^{th} Lucas number) have occurred in questions in cryptography [3] and in the study of periodic points in algebraic topology [11]. They are also the subject of study in [5]. We describe here a simple method for finding results concerning the Zeckendorf decomposition of such a product. We let $\beta = (1+\sqrt{5})/2$ throughout the paper, and we make use of the connection between the β-expansion and the Zeckendorf decomposition as developed by Grabner et al. in [8] and [9].

The β-expansion of $n \in \mathbb{N}$ is the unique finite sum of integral powers of β that equals n and contains no consecutive powers of β. Grabner et al., in [8] and [9], prove that for m sufficiently large the Zeckendorf decomposition of kF_m can be produced by replacing β^i in the β-expansion of k with F_{m+i}. For example, the β-expansion of 5 is $\beta^3 + \beta^{-1} + \beta^{-4}$, and the Zeckendorf decomposition of $5F_{10}$ is $F_{13} + F_9 + F_6$. See [1], [2], [6], [10], [14], and Section 2 for background on the β-expansion.

We have found that by studying short lists of β-expansions of small positive integers we can easily observe patterns that represent new results. In Section 4 we improve upon the results of [5] involving the number of addends in the Zeckendorf decomposition of mF_m and we include a proof of Conjecture 3 from the same paper. This conjecture states that, for certain values of m and k, the Zeckendorf decomposition of $(mL_{2k}+1)(F_{mL_{2k}+1})$ contains $F_{mL_{2k}+1}$ as one of its terms. This is equivalent to saying that β^0 occurs in the β-expansion of $mL_{2k}+1$. Most of the identities in [5] can be discovered easily using the techniques given here, as we demonstrate in Section 3. While a computer can be used to form lists of β-expansions, we were able to discover all the results in Sections 3 and 4 easily by hand. All proofs are provided in Section 6.

The developments presented here provide the background necessary for [12], joint work with L. Sanchis, in which we prove Conjecture 1 from [5]. The conjecture involves the ratio of natural numbers k that do not have F_k in the Zeckendorf decomposition of kF_k to those natural numbers that do. The list of β-expansions of k for $1 \leq k \leq 500$, produced easily by a computer, was sufficient to allow us to discover the recursive patterns in the β-expansions and then to prove that the conjecture is correct. This result also answers an equivalent question posed by Bergman in [1] concerning the frequency of positive integers n with β^0 appearing in the β-expansion of n.

We present an algorithm for finding the β-expansion of a positive integer that can be used to efficiently produce a list of β-expansions. The beginning of this list is given in Section 2. The algorithm actually applies more generally. Given a sum $n = \sum_{i=m}^{M} \lambda_1 F_i$ with $m, M \in \mathbb{Z}$ and $\lambda_i \in \mathbb{N}$

for all i, the algorithm produces a representation of n as a sum of nonconsecutive Fibonacci numbers, some of which may have negative indices. If the smallest index in the resulting sum is at least 2, then the algorithm has produced the Zeckendorf decomposition of n without requiring the calculation of the value of n. This algorithm runs in time that is linear in $M-m+\sum_{i=m}^{M}\lambda_i$. For another algorithm that produces the Zeckendorf decomposition of n with the same input (but does not give the β-expansion of a number) see, for example, [7].

2. PRELIMINARIES

Remark 2.1: Note that in [8] and [9] the indices for Fibonacci and Lucas numbers are different from the standard used here. We use $F_0=0, F_1=1, L_0=2$, and $L_1=1$. For $x<0$, let F_x be equal to $(-1)^{-x+1}F_{-x}$.

Definition 2.2: Let $n\in\mathbb{N}$. The Zeckendorf decomposition of n is the unique expression of n as a sum of Fibonacci numbers of the form $\sum_{i=2}^{r}\mu_iF_i$, with $r\in\mathbb{N}$, $\mu_i\in\{0,1\}$, and with $\mu_i\mu_{i+1}=0$.

Definition 2.3: Let β be the golden ratio $(1+\sqrt{5})/2$. For any $n\in\mathbb{N}$, the β-expansion of n is the unique expression of n as a finite sum of integral powers of β with no consecutive powers occurring. That is, $n=\sum_{i=-\infty}^{\infty}e_i\beta^i$ with $e_i\in\{0,1\}$, $e_ie_{i+1}=0$, and with at most finitely many e_i equal to one.

For this value of β, the β-expansion was first defined by Bergman in 1957 in [1]. For generalizations using other values of β, see, for example, [2], [6], [14], and [15].

Definition 2.4: For $k\in\mathbb{N}$, the lower width of k, $\ell(k)$ [resp. the upper width of k, $u(k)$] is defined to be the absolute value of the smallest (resp. largest) exponent that appears in the β-expansion of k.

For example, the β-expansion of 12 is $\beta^{-6}+\beta^{-3}+\beta^{-1}+\beta^5$, so $\ell(12)=6$ and $u(12)=5$.

The following is a restatement of Lemma 1 and Theorem 1 in [9] for the special case of Fibonacci numbers. See also Theorem 1 in [8].

Theorem 2.5 (Grabner et al. [8]): For $k\in\mathbb{N}$ and for $n\geq\ell(k)+2$, if the β-expansion of k is $\sum_{i=-\ell(k)}^{u(k)}e_i\beta^i$, then the Zeckendorf decomposition of kF_n is $\sum_{i=-\ell(k)}^{u(k)}e_iF_{i+n}$. For $k\in\mathbb{N}$, we have that $\ell(k)$ is the even number defined by $L_{\ell(k)-1}<k\leq L_{\ell(k)+1}$. If $2<k<L_{\ell(k)}$, then $u(k)=\ell(k)-1$. If $k\geq L_{\ell(k)}$, then $u(k)=\ell(k)$. We also have that $u(1)=0$ and $u(2)=1$.

For example, the β-expansion of 10 is $\beta^{-4}+\beta^{-2}+\beta^2+\beta^4$, as can be determined quickly by the algorithm of Section 5 (see 5.7), and the Zeckendorf decomposition of $10F_{5000}$ is $F_{4996}+F_{4998}+F_{5002}+F_{5004}$. The power of Theorem 2.10 in [8] is clear here. Using the greedy algorithm, we would have needed to calculate the value of $10F_{5000}$, which is daunting.

As usual, a sum of Fibonacci numbers will be represented by a vector of zeros and ones. A one occurs in coordinate s if F_s appears in the sum. We allow negative indices.

Definition 2.6: We define V to be the infinite dimensional vector space over $\mathbb{Z}$ given by

$$V:=\{(\ldots,v_{-1},v_0,v_1,\underline{v_2},v_3,\ldots):v_i\in\mathbb{Z}\,\forall_i,\text{ with at most finitely many }v_i\text{ nonzero}\}.$$

For convenience, we underline the second coordinate. We define V^+ to be the subset of V that consists of all vectors of V with all entries nonnegative.

All vectors in V are infinite dimensional, but we will abuse notation and omit the entries before the first possibly nonzero entry and after the last possibly nonzero entry. If the entries are all single digits, we may omit commas and parentheses.

Definition 2.7: Let $n \in \mathbb{N}$. Let $\vec{z}(n)$ be the vector in V^+ corresponding to the Zeckendorf decomposition of n that has 0 in the first coordinate.

In Definition 2.7, we must require that the vector have zero in the first coordinate in order to have $\vec{z}$ well defined. For example, the Zeckendorf decomposition of 4 is $1+3$, which can be represented by either F_1+F_4 or F_2+F_4. Whenever 1 occurs in the Zeckendorf decomposition, we always represent it as F_2 in the image of $\vec{z}$. Thus, $\vec{z}(4) = (0, \underline{1}, 0, 1)$.

Definition 2.8: We define the function $\vec{\beta}: \mathbb{N} \to V^+$ so that $\vec{\beta}(n)$ is the vector in V^+ with $v_i = e_{i-2}$ when the β-expansion of n is $\sum_{i=-\infty}^{\infty} e_i \beta^i$. Thus, the coefficient of β^0 is underlined.

For example, $\vec{\beta}(12)$ is represented by $100101\underline{0}00001$. Here the exponents of β increase from left to right, which does not match the usual notation for a β-expansion. We must choose between the usual notation for $\vec{z}(n)$ and for $\vec{\beta}(n)$. Because this paper concerns Zeckendorf decompositions, we have chosen the former.

The β-expansion of k is as follows for $1 \le k \le 20$, with the exponents of β increasing from left to right.

k	$\vec{\beta}(k)$												
1							$\underline{1}$						
2					1	0	$\underline{0}$	1					
3					1	0	$\underline{0}$	0	1				
4					1	0	$\underline{1}$	0	1				
5			1	0	0	1	$\underline{0}$	0	0	1			
6			1	0	0	0	$\underline{0}$	1	0	1			
7			1	0	0	0	$\underline{0}$	0	0	0	1		
8			1	0	0	0	$\underline{1}$	0	0	0	1		
9			1	0	1	0	$\underline{0}$	1	0	0	1		
10			1	0	1	0	$\underline{0}$	0	1	0	1		
11			1	0	1	0	$\underline{1}$	0	1	0	1		
12	1	0	0	1	0	1	$\underline{0}$	0	0	0	0	1	
13	1	0	0	1	0	0	$\underline{0}$	1	0	0	0	1	
14	1	0	0	1	0	0	$\underline{0}$	0	1	0	0	1	
15	1	0	0	1	0	0	$\underline{1}$	0	1	0	0	1	
16	1	0	0	0	0	1	$\underline{0}$	0	0	1	0	1	
17	1	0	0	0	0	0	$\underline{0}$	1	0	1	0	1	
18	1	0	0	0	0	0	$\underline{0}$	0	0	0	0	0	1
19	1	0	0	0	0	0	$\underline{1}$	0	0	0	0	0	1
20	1	0	0	0	1	0	$\underline{0}$	1	0	0	0	0	1

It is possible to generate the k^{th} row in this list by applying the algorithm developed in Section 5 to the vector $(0, \underline{k}, 0)$ (see 5.5). We will see in Remark 5.9 that we may instead move from one row to the next by adding one to the underlined entry and applying $\mathscr{A}$ to the result. This second method is much more efficient.

Definition 2.9: We define a linear transformation that shifts the entries of a vector. For $t \in \mathbb{Z}$, let $s_t : V \to V$ be given by the following. If $\vec{v} \in V$ has coordinates v_i for $i \in \mathbb{Z}$, then $s_t(\vec{v})$ is the vector with coordinates $w_i := v_{i-t}$.

Next, we restate part of Theorem 2.5 using the notation of this section.

Theorem 2.10 (Grabner et al. [8]): For $k \in \mathbb{N}$ and for $n \in \mathbb{Z}$, kF_n is represented by the vector $s_{n-2}(\vec{\beta}(k))$. For $n \geq \ell(k)+2$, this vector is $\vec{z}(kF_n)$.

3. PATTERNS IN β-EXPANSIONS

Most of the identities in [5] can be found using lists of β-expansions. For example, (2.4) of [5] gives the Zeckendorf decomposition for $4F_kF_{n+k}$ whenever $k, n \geq 3$. This could be determined by adding the formulas given in [5] for F_kF_{n+k} and for $3F_kF_{n+k}$ and reducing the result so that no two consecutive Fibonacci numbers occur. On the other hand, we are able to determine the pattern for the β-expansions of $4F_k$ from scratch very quickly by considering the list below. We then arrive at the Zeckendorf decomposition of $4F_kF_{n+k}$ by simply shifting the vector $\vec{\beta}(4F_k)$ to the right by $n+k-2$ spaces.

The following list provides the β-expansion for $4F_k$ as k increases from 3 to 10. Note that we can add two consecutive rows in this list and then apply the algorithm from Section 5 to the sum. The result will be the next row in the list. This is easy to do by hand. The diagonal lines of ones appear in a predictable pattern that will continue, as can be proven by induction.

```
k = 3 :                  1 0 0 0 1 0 0 0 1
k = 4 :              1 0 0 1 0 1 0 0 0 0 0 1
k = 5 :              1 0 0 0 1 0 0 1 0 0 0 0 1
k = 6 :          1 0 0 1 0 1 0 0 0 0 1 0 0 0 0 1
k = 7 :          1 0 0 0 1 0 0 1 0 0 0 1 0 0 0 0 1
k = 8 :      1 0 0 1 0 1 0 0 0 0 1 0 0 0 1 0 0 0 0 1
k = 9 :      1 0 0 0 1 0 0 1 0 0 0 1 0 0 0 1 0 0 0 0 1
k = 10 : 1 0 0 1 0 1 0 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 0 1
```

For $n \geq 3$ and $k \geq 3$ we have, as in (2.4) of [5],

$$4F_kF_{n+k} = \begin{cases} F_{n-2} + F_{n+1} + F_{n+3} + F_{2k+n+1} + \sum_{j=1}^{(k-4)/2} F_{4j+n+4} & (k \text{ even}), \\ F_{n-1} + F_{n+3} + F_{2k+n+1} + \sum_{j=1}^{(k-3)/2} F_{4j+n+2} & (k \text{ odd}). \end{cases}$$

Note that a similar method can be used for finding the β-expansion of mL_k and hence for finding the Zeckendorf decomposition of products of the form mL_kF_{n+k}.

4. NEW RESULTS

We summarize here the new results we have found using the β-expansion. Proofs are provided in Section 6. We begin with a technical definition and then state precisely in 4.2 the useful fact that, if two Zeckendorf decompositions have indices that do not overlap significantly, then the two Zeckendorf decompositions can meld into the Zeckendorf decomposition of the sum.

Definition 4.1: For $x \in \mathbb{Z}$, we say that a $\vec{v} \in V^+$ is reduced to index x if every entry with index (i.e., coordinate) $\geq x$ is either zero or one and $\vec{v}$ has at most finitely many ones with index $\geq x$ and $\vec{v}$ has no consecutive ones with indices $\geq x$. If $\vec{v}$ is reduced for all $x \in \mathbb{Z}$, then $\vec{v}$ is totally reduced.

Note that, for all $n \in \mathbb{N}$, the vectors $\vec{z}(n)$ and $\vec{\beta}(n)$ are totally reduced.

Fact 4.2: If $n, m \in \mathbb{N}$ and if $\vec{z}(n)+\vec{z}(m)$ is totally reduced, then $\vec{z}(n+m)=\vec{z}(n)+\vec{z}(m)$. The same is true for β-expansions by 2.10. One way to determine whether $\vec{z}(n)+\vec{z}(m)$ is totally reduced is to consider the following two sets. Let $I_n=\{i \in \mathbb{Z}: F_i$ is in the Zeckendorf decomposition of $n\}$, and let I_m be the corresponding set for m. Let $d=\min\{|i-j|: i \in I_n, j \in I_m\}$. Then $\vec{z}(n)+\vec{z}(m)$ is totally reduced if $d \geq 2$.

Let $Q(n)=nF_n$, and note that we will find it useful to use exponents in vectors. For example, the β-expansion of L_8 is given by $10000000\underline{0}0000001$, which we may write as $10^7\underline{0}0^7 1$. Similarly, the β-expansion of L_9 is $10101010\underline{1}01010101$, which we may write as $(10)^4\underline{1}(01)^4$.

Proposition 4.3: For $k \geq 2$, we have $\vec{\beta}(2L_{2k})=110010^{2k-2}\underline{0}0^{2k-3}1001$. Thus, the Zeckendorf decomposition of $Q(2L_{2k})=F_{2k+1+2L_{2k}}+F_{2k-2+2L_{2k}}+F_{-2k-2+2L_{2k}}+F_{-2k+1+2L_{2k}}$.

The preceding proposition is proven in detail in Section 6, but to give an idea of the flavor of such proofs, we provide a sketch here. We have $\vec{\beta}(L_{2k})=10^{2k-1}\underline{0}0^{2k-1}1$ (see (1.5) of [4] and apply 2.10). We think of $2L_{2k}$ as $20^{2k-1}\underline{0}0^{2k-1}2$ and we prove that $\vec{\beta}(2L_{2k})$ is given by

$$\overbrace{1001}0^{2k-2}\underline{0}0^{2k-3}\overbrace{1001},$$

where the braces mark the vectors $s_{-2k}(\vec{\beta}(2))$ and $s_{2k}(\vec{\beta}(2))$. Because the two braces do not touch, the entire vector is totally reduced.

In the following propositions, let $f[n]$ denote the number of addends in the Zeckendorf decomposition of n as in [5]. Note that $f[Q(m)]$ is equal to the number of ones in the vector $\vec{\beta}(m)$ by 2.10. The next two propositions are generalizations of (3.3) and (3.4) in [5].

Proposition 4.4: If $k \geq 2$, and if $1 \leq m \leq L_{2k-1}$, then $f[Q(L_{2k}+m)]=2+f[Q(m)] \leq 2k+1$. Moreover, $\vec{z}(Q(L_{2k}+m))=\vec{z}(L_{2k}F_{L_{2k}+m})+\vec{z}(mF_{L_{2k}+m})$.

Proposition 4.5: If $k \geq 2$, and if $1 \leq m \leq L_{2k-3}$, then $f[Q(2L_{2k}+m)]=4+f[Q(m)]$. Moreover, $\vec{z}(Q(2L_{2k}+m))=\vec{z}(2L_{2k}F_{2L_{2k}+m})+\vec{z}(mF_{2L_{2k}+m})$.

In [5] a positive integer n is said to have Property $\mathcal{P}$ if F_n occurs in the Zeckendorf decomposition of nF_n. This is equivalent to stating that a one occurs in the underlined coordinate of $\vec{\beta}(n)$. We prove Conjecture 3 of [5] in the following proposition.

Proposition 4.6: If $m, k \in \mathbb{N}$ with $1 \leq k$ and $1 \leq m \leq L_{2k-1}$, then mL_{2k} does not have Property $\mathcal{P}$, and $mL_{2k}+1$ does have Property $\mathcal{P}$.

Proposition 4.7: For $k \geq 2$, we have $\vec{\beta}(L_{2k+1}+L_{2k-1})=100100(10)^{k-2}\underline{1}(01)^{k-1}001$. Thus, we see that $Q(L_{2k+1}+L_{2k-1})$ has Property $\mathcal{P}$.

See Section 6 for proofs of the propositions in this section.

5. THE ALGORITHM

The algorithm begins with a positive integer n expressed as $n = \sum_{i=m}^{M} \lambda_i F_i$, with λ_i any nonnegative integer for $i \in \mathbb{Z}$, and with $m, M \in \mathbb{Z}$. It ends with an expression for n as a sum of Fibonacci numbers with nonconsecutive (possibly negative) indices. This sum is the Zeckendorf decomposition for n under certain conditions. There are other algorithms that produce Zeckendorf decompositions (normal forms) in this setting (see [7]). The advantage of the algorithm given here is that it allows us to find the β-expansion of $k \in \mathbb{N}$ by applying the algorithm to $(0, \underline{k}, 0)$ (see 5.5).

Definition 5.1: Let $\vec{v} \in V$ be a vector with coordinates v_i for $i \in \mathbb{Z}$. Let $\sigma: V \to \mathbb{Z}$ be the function given by $\sigma(\vec{v}) := \sum_{i=-\infty}^{\infty} v_i F_i$. Note that σ is a linear function and that it is not injective.

Verbose Description of the Algorithm: We begin with n represented by the vector $\vec{v} := (\ldots, \lambda_0, \lambda_1, \underline{\lambda_2}, \lambda_3, \ldots)$, where $\lambda_i = 0$ for $i > M$ and for $i < m$ as above. Thus, the initial values for the entries in $\vec{v}$ are $v_i = \lambda_i$ for $i \in \mathbb{Z}$. First, we search for the smallest integer x for which the vector $\vec{v}$ is reduced to index x. If there is no such integer, then we are done. Details of the search are below in the second description of the algorithm. We assign $t := x-1$ if $v_x = 0$ and $t := x$ if $v_x = 1$. Note that this implies that $v_{t+1} = 0$ and $v_t \geq 1$.

Case 1. $v_{t-1} \neq 0$. We have $(\ldots, v_{t-1}, v_t, 0, \ldots)$. We replace $v_{t-1}, v_t, 0$ with $v_{t-1}-1,\ v_t - 1,\ 1$. This does not change the value of $\sigma(\vec{v})$, because $F_t + F_{t-1} = F_{t+1}$. We return to the beginning of the algorithm and search for a new value of x.

Case 2. We have $(\ldots, v_{t-2}, 0, v_t, 0, \ldots)$, and, because the vector is not reduced to index $t-1$, $v_t > 1$. We replace $v_{t-2}, 0, v_t, 0$ with $v_{t-2}+1, 0, v_t - 2, 1$. This does not change the value of $\sigma(\vec{v})$. To see this, consider two smaller steps. We can replace $v_{t-2}, 0, v_t, 0$ with $v_{t-2}+1, 1, v_t - 1, 0$ because $F_t = F_{t-1} + F_{t-2}$. Now we have two consecutive nonzero entries, so we can do as in the first case. This results in $v_{t-2}+1, 0, v_t - 2, 1$. Note that the sum of all the entries in the vector $\vec{v}$ has not changed. We return to the beginning of the algorithm.

As stated above, the algorithm terminates when there is no minimal value x.

Definition 5.2: Let $\mathcal{A}: V^+ \to V^+$ be the function that assigns to a vector $\vec{v} \in V^+$ the result of applying this algorithm to $\vec{v}$.

Precise Description of the Algorithm: As above, $n = \sum_{i=m}^{M} \lambda_i F_i$.

```
max := M, min := m;
t := max;
while (t ≥ min) do {
    if (v_t = 0) then t := t − 1;
    else if (v_{t−1} = 0 and v_t = 1) then t := t − 2;
    else if (v_{t−1} ≠ 0) then {
        v_{t+1} := 1; v_t := v_t − 1; v_{t−1} := v_{t−1} − 1;
        if (v_{t+2} = 0) then t := t + 1;
        else t := t + 2;
    }
```

```
    else {
        v_{t+1} := 1; v_t := v_t - 2; v_{t-2} := v_{t-2} + 1;
        if (t - 2 < min) then min := t - 2;
        if (v_{t+2} = 0) then t := t + 1;
        else t := t + 2;
    }
}
```

Remark 5.3: The algorithm $\mathcal{A}$ is designed so that, for all $\vec{v} \in V^+, \sigma(\vec{v}) = \sigma(\mathcal{A}(\vec{v}))$, and $s_t(\mathcal{A}(\vec{v})) = \mathcal{A}(s_t(\vec{v}))$ for all $t \in \mathbb{Z}$. The second equality follows from the fact that the algorithm is independent of the numbering of the coordinates of the vector $\vec{v}$.

Proofs of the results from this section are postponed until Section 6.

Proposition 5.4: The algorithm terminates in a finite number of steps for any vector $\vec{v} \in V^+$. The result $\mathcal{A}(\vec{v})$ is totally reduced.

Proposition 5.5: For all $k \in \mathbb{N}$, $\vec{\beta}(k) = \mathcal{A}(0, \underline{k}, 0)$.

Remark 5.6: If $\vec{v} \in V^+$, and if $\mathcal{A}(\vec{v})$ has no nonzero entries for all coordinates with index less than 2, then $\mathcal{A}(\vec{v}) = \vec{z}(\sigma(\vec{v}))$. For $k \in \mathbb{N}$ and $n \geq \ell(k)+2$, we have $\vec{z}(kF_n) = s_{n-2}(\vec{\beta}(k))$ as in Theorem 2.10.

Example 5.7: We apply the algorithm to $10F_2$ to find the β-expansion of 10.

				$\underline{10}$					
		1	0	$\underline{8}$	1				
		1	0	$\underline{7}$	0	1			
		2	0	$\underline{5}$	1	1			
		2	0	$\underline{5}$	0	0	1		
		3	0	$\underline{3}$	1	0	1		
		3	0	$\underline{2}$	0	1	1		
		3	0	$\underline{2}$	0	0	0	1	
		4	0	$\underline{0}$	1	0	0	1	
1	0	2	1	$\underline{0}$	1	0	0	1	
1	0	1	0	$\underline{1}$	1	0	0	1	
1	0	1	0	$\underline{0}$	0	1	0	1	$= \vec{\beta}(10)$

Note that in the 9th row we have in coordinates 2 through 6 the Zeckendorf decomposition of 10, with a 4 in the 0th coordinate. A similar pattern occurs whenever this method is used to find the β-expansion of any positive integer.

Having determined the β-expansion of 10, we can apply Theorem 2.10 and see that $\vec{z}(10F_{5000}) = s_{4998}(10100\underline{0}101)$. This is much easier than calculating the value of $10F_{5000}$ and applying the greedy algorithm.

Theorem 5.8: For $\vec{v}, \vec{w} \in V^+$ and for $k \in \mathbb{N}$, we have $\mathcal{A}(\mathcal{A}(\vec{v}) + \vec{w}) = \mathcal{A}(\vec{v} + \vec{w})$ and $\mathcal{A}(k\vec{v}) = \mathcal{A}(k\mathcal{A}(\vec{v}))$. In addition, for all $n, m \in \mathbb{N}$, we have $\vec{\beta}(nm) = \mathcal{A}(n\vec{\beta}(m))$ and $\vec{\beta}(n+m) = \mathcal{A}(\vec{\beta}(n) + \vec{\beta}(m))$.

Remark 5.9: The list of β-expansions in Section 2 above can be generated by applying the algorithm to $(0, \underline{k}, 0)$ for each k (see 5.5). Theorem 5.8 provides a more efficient method for deriving the list. Once we have found that $\vec{\beta}(2) = 10\underline{0}1$, we note that

$$\vec{\beta}(3) = \mathcal{A}(0, \underline{3}, 0) = \mathcal{A}(\mathcal{A}(0, \underline{2}, 0) + (0, \underline{1}, 0)) = \mathcal{A}((\vec{\beta}(2)) + (0, \underline{1}, 0)).$$

To move from the β-expansion of $k-1$ to that of k, we need only add one to the underlined entry (which corresponds to β^0) and then apply the algorithm.

6. PROOFS

Lemma 6.1: If a vector $\vec{v} \in V^+$ is reduced to index $s+1$, and if $v_s = 1$, then, when the algorithm is applied, none of the entries with index less than s will be changed until after the algorithm has changed $\vec{v}$ into a vector that is reduced to index s.

Proof of 6.1: We induct upon n using the following induction hypothesis:

If $\vec{v}$ is reduced to index $t+1$ for some t with exactly n nonzero entries (ones) with index greater than t, and if $v_t = 1$, then none of the entries with index less than t will be changed until after the algorithm has changed $\vec{v}$ into a vector that is reduced to index t.

Suppose $n = 1$. Then either $v_s = 1$ and $v_{s+1} = 0$, which means that $\vec{v}$ is already reduced to index s, or $v_s = 1 = v_{s+1}$ and $v_{s+i} = 0$ for $i \geq 2$, which means that the algorithm will change the vector so that $v_s = 0 = v_{s+1}$ and $v_{s+2} = 1$ without changing any other entries. The new vector is reduced to index s. Thus, the statement is true for $n = 1$.

Now induct on n. Consider the triple $1, v_{s+1}, v_{s+2}$. If this triple is $1, 0, 0$ or $1, 0, 1$, then $\vec{v}$ is already reduced to index s. If the triple is $1, 1, 0$, the algorithm first replaces the triple with $0, 0, 1$, and we can use the inductive hypothesis. We now have a vector that is reduced to index $s+3$ that has $v_{s+2} = 1$. The number of ones with index greater than $s+2$ is one smaller than the number of ones we had originally with index greater than s. Thus, the algorithm does not change the values of entries with index less than $s+2$ until the vector has been changed to a new vector that is reduced to index $s+2$. This means that we will have the triple $0, 0, 1$ either unchanged or replaced with $0, 0, 0$. In either case, the resulting vector is reduced to index s.

Proof of 5.4: If $\vec{v} \in V^+$ is reduced to index s for all s, then the algorithm does not ever change the vector. We have $\mathcal{A}(\vec{v}) = \vec{v}$, and the proposition is proven for that case.

Otherwise, there is a unique $x(\vec{v}) \in \mathbb{Z}$ with $\vec{v}$ reduced to index $x(\vec{v})$ and with $\vec{v}$ not reduced to index $x(\vec{v})-1$. In this case, we define $r(\vec{v})$ to be the sum of all entries of $\vec{v}$ with index less than $x(\vec{v})$. We will see that $r(\vec{v})$ will reach zero in a finite number of steps. This means that the algorithm stops in a finite number of steps and that the vector $\mathcal{A}(\vec{v})$ is in the desired form.

We refer now to the cases given in the *Verbose Description of the Algorithm* in Section 5. The algorithm first assigns $t := x(\vec{v}) - 1$ if $v_x = 0$ and assigns $t := x(\vec{v})$ if $v_x = 1$.

In *Case 1*, the triple $(v_{t-1}, v_t, 0)$ is replaced by $(v_{t-1} - 1, v_t - 1, 1)$ and the new vector is reduced to index $t+2$ with $v_{t+1} = 1$. By Lemma 6.1, we know that the algorithm will next change the vector so that it is reduced to index $t+1$ without changing the values of entries with index less than $t+1$. At this stage, the new vector $\vec{v}$ has a new $x(\vec{v})$-value that is less than or equal to $t+1$. Thus, the new value of $r(\vec{v})$ is at most 2 less than the old value of $r(\vec{v})$.

In *Case 2*, we see that $(v_{t-2}, 0, v_t, 0)$ is replaced by $(v_{t-2}+1, 0, v_t-2, 1)$ and the new vector is reduced to index $t+2$ with $v_{t+1}=1$. By Lemma 6.1, we know that the algorithm will next change the vector so that it is reduced to index $t+1$ without changing the values of entries with index less than $t+1$. At this stage, the new vector $\vec{v}$ has a new $x(\vec{v})$-value that is less than or equal to $t+1$. Thus, the new value of $r(\vec{v})$ is at most 1 less than the old value of $r(\vec{v})$.

In both cases, the value of $r(\vec{v})$ decreases. Thus, the algorithm terminates in a finite number of steps. The vector that results will be reduced to index s for all $s \in \mathbb{Z}$.

Proof of 5.5: This result follows from the work of Grabner et al., but a direct proof is as follows. Because $\beta^i + \beta^{i+1} = \beta^{i+2}$ for all $i \in \mathbb{Z}$, we can replace each F_i in the description of the algorithm with β^{i-2} and replace each σ with σ', where $\sigma'(\vec{v}) = \sum_{i=-\infty}^{\infty} v_i \beta^{i-2}$. Because $kF_2 = k\beta^0$, for the vector $(0, \underline{k}, 0)$ the algorithm will produce the same result either way. Thus, $\mathcal{A}(0, \underline{k}, 0)$ is $\vec{\beta}(k)$.

Proof of 5.8: Let $\vec{x} = \mathcal{A}(\vec{v}) + \vec{w}$, and let $\vec{y} = \vec{v} + \vec{w}$. We first prove that, for all $t \in \mathbb{Z}$, we have $\sigma(s_t(\vec{x})) = \sigma(s_t(\vec{y}))$. We have, using Remark 5.3 and the fact that s_t and σ are linear, $\sigma(s_t(\vec{x})) = \sigma(s_t\mathcal{A}(\vec{v})) + \sigma(s_t(\vec{w})) = \sigma(s_t\mathcal{A}(\vec{v})) + \sigma(s_t(\vec{w})) = \sigma(s_t(\vec{v})) + \sigma(s_t(\vec{w})) = \sigma(s_t(\vec{y}))$.

Next we prove that, for all $t \in \mathbb{Z}$ and for all $k \in \mathbb{N}$, we have $\sigma(s_t(k\vec{v})) = \sigma(s_t(k\mathcal{A}\vec{v}))$. We have $\sigma(s_t(k\vec{v})) = k\sigma(s_t(\vec{v})) = k\sigma(\mathcal{A}(s_t(\vec{v}))) = k\sigma(s_t(\mathcal{A}\vec{v})) = \sigma(s_t(k\mathcal{A}\vec{v}))$.

Theorem 2.10 implies the following. There exist $t_1, t_2, t_3, t_4 \in \mathbb{N}$ such that, for all $i \geq 0$, we have $\mathcal{A}(s_{t_1+i}(\vec{x})) = \vec{z}(\sigma(s_{t_1+i}(\vec{x})))$, $\mathcal{A}(s_{t_2+i}(\vec{y})) = \vec{z}(\sigma(s_{t_2+i}(\vec{y})))$, $\mathcal{A}(s_{t_3+i}(k\vec{v})) = \vec{z}(\sigma(s_{t_3+i}(k\vec{v})))$, and $\mathcal{A}(s_{t_4+i}(k\mathcal{A}\vec{v})) = \vec{z}(\sigma(s_{t_4+i}(k\mathcal{A}\vec{v})))$. Let $t = \max\{t_1, t_2, t_3, t_4\}$.

Using 5.3 again, we see that $s_t(\mathcal{A}(k\vec{v})) = \mathcal{A}(s_t(k\vec{v})) = \vec{z}(\sigma(s_t(k\mathcal{A}\vec{v})))$, and $\vec{z}(\sigma(s_t(k\mathcal{A}\vec{v}))) = \mathcal{A}(s_t(k\mathcal{A}\vec{v})) = s_t(\mathcal{A}(k\mathcal{A}\vec{v}))$. But s_t is one-to-one. Thus, $\mathcal{A}(k\vec{v}) = \mathcal{A}(k\mathcal{A}(\vec{v}))$.

Similarly, $s_t(\mathcal{A}(\vec{x})) = \mathcal{A}(s_t(\vec{x})) = \vec{z}(\sigma(s_t(\vec{x}))) = \vec{z}(\sigma(s_t(\vec{y}))) = \mathcal{A}(s_t(\vec{y})) = s_t(\mathcal{A}(\vec{y}))$. Again because s_t is one-to-one, $\mathcal{A}(\vec{x}) = \mathcal{A}(\vec{y})$. Thus, $\mathcal{A}(\mathcal{A}(\vec{v}) + \vec{w}) = \mathcal{A}(\vec{v} + \vec{w})$.

Next, let $m, n \in \mathbb{N}$. We have that $\vec{\beta}(nm) = \mathcal{A}(0, \underline{nm}, 0) = \mathcal{A}(n(0, \underline{m}, 0)) = \mathcal{A}(n\mathcal{A}(0, \underline{m}, 0)) = \mathcal{A}(n\vec{\beta}(m))$, and also, $\vec{\beta}(nm) = \mathcal{A}(0, \underline{n+m}, 0) = \mathcal{A}((0, \underline{n}, 0) + (0, \underline{m}, 0)) = \mathcal{A}(\mathcal{A}(0, \underline{n}, 0) + \mathcal{A}(0, \underline{m}, 0)) = \mathcal{A}(\vec{\beta}(n) + \vec{\beta}(m))$. Thus, Theorem 5.8 is proven.

Proof of 4.3: We have, for all $k \in \mathbb{N}$, that $\vec{\beta}(L_{2k}) = 10^{2k-1}\underline{0}0^{2k-1}1$ (see Proposition 10 of [5] and apply 2.10). Thus, using Theorem 5.8, we have $\vec{\beta}(2L_{2k}) = \mathcal{A}(2\vec{\beta}(L_{2k})) = \mathcal{A}(20^{2k-1}\underline{0}0^{2k-1}2)$ $= \mathcal{A}(s_{-2k}(0, \underline{2}, 0) + s_{2k}(0, \underline{2}, 0)) = \mathcal{A}(\mathcal{A}s_{-2k}(0, \underline{2}, 0) + \mathcal{A}s_{2k}(0, \underline{2}, 0)) = \mathcal{A}(s_{-2k}\mathcal{A}(0, \underline{2}, 0) + s_{2k}\mathcal{A}(0, \underline{2},)) = \mathcal{A}(s_{-2k}\vec{\beta}(2) + s_{2k}\vec{\beta}(2)) = \mathcal{A}((10010^{2k-2}\underline{0}) + (\underline{0}0^{2k-3}1001)) = 10010^{2k-2}\underline{0}0^{2k-3}1001$. Finally, we apply 2.10 to complete the proof.

Proof of 4.4: We have that $\vec{\beta}(L_{2k}) = 10^{2k-1}\underline{0}0^{2k-1}1$, as in the proof of Theorem 4.3. Because $m \leq L_{2k-1}$, we have by 2.5 that $\ell(m), u(m) \leq 2k-2$. Thus, using 4.2, we see that $\vec{\beta}(L_{2k}+m) = \vec{\beta}(L_{2k}) + \vec{\beta}(m)$. Thus, $f[Q(L_{2k}+m)]$ is the number of ones in $\vec{\beta}(L_{2k})$ plus the number of ones in $\vec{\beta}(m)$, and $f[Q(L_{2k}+m)] = 2 + f[Q(m)]$. Because $\ell(L_{2k}) = u(L_{2k}) = 2k$, there can be at most $2k+1$ addends in $Q(m)$. This proves the last inequality.

Proof of 4.5: Using 4.3, we have that $\vec{\beta}(2L_{2k}) = 10010^{2k-2}\underline{0}0^{2k-3}1001$. Because $m \le L_{2k-3}$, we have by 2.5 that $\ell(m), u(m) \le 2k-4$. Using 4.2, we see that $\vec{\beta}(2L_{2k}+m) = \vec{\beta}(2L_{2k}) + \vec{\beta}(m)$. Thus, $f[Q(2L_{2k}+m)]$ is the number of ones in $\vec{\beta}(2L_{2k})$ plus the number of ones in $\vec{\beta}(m)$, and $f[Q(2L_{2k}+m)] = 4 + f[Q(m)]$.

Proof of 4.6: We have $\vec{\beta}(L_{2k}) = 10^{2k-1}\underline{0}0^{2k-1}1$, as in the proof of 4.3. Let $\vec{v} = s_{-2k}(0\underline{1}0)$, and let $s_{2k}(0\underline{1}0)$. Then $\vec{\beta}(L_{2k}) = \vec{v} + \vec{w}$, and so by Theorem 5.8 we have $\vec{\beta}(mL_{2k}) = \mathcal{A}(m\vec{v} + m\vec{w}) = \mathcal{A}(\mathcal{A}(m\vec{v}) + \mathcal{A}(m\vec{w})) = \mathcal{A}(\mathcal{A}(s_{-2k}(0,\underline{m},0)) + \mathcal{A}(s_{2k}(0,\underline{m},0))) = \mathcal{A}(s_{-2k}(\mathcal{A}(0,\underline{m},0)) + s_{2k}(\mathcal{A}(0,\underline{m},0)))$ $= \mathcal{A}(s_{-2k}(\vec{\beta}(m)) + s_{2k}(\vec{\beta}(m)))$. By Fact 4.2 and Theorem 2.5, $s_{-2k}(\vec{\beta}(m)) + s_{2k}(\vec{\beta}(m))$ is totally reduced whenever $m \le L_{2k-1}$. Thus, $\vec{\beta}(mL_{2k}) = s_{-2k}(\vec{\beta}(m)) + s_{2k}(\vec{\beta}(m))$ for $1 \le m \le L_{2k-1}$. Whenever $m \le L_{2k-1}$, the two shifted β-expansions of m will not overlap, and in fact there will be zeros in the coordinates corresponding to β^{-1}, β^0, and β^1. Thus, mL_{2k} does not have Property $\mathcal{P}$. When a one is inserted in the coordinate corresponding to β^0, the resulting vector is totally reduced and equals $\vec{\beta}(mL_{2k}+1)$ (see 5.9). Thus, $mL_{2k}+1$ does have Property $\mathcal{P}$.

Proof of 4.7: We have, for all $k \in \mathbb{N}$, that $\vec{\beta}(L_{2k+1}) = (10)^k\underline{1}(01)^k$ (see (3.1) of [5]). Thus, using 5.8, we have $\vec{\beta}(L_5 + L_3) = \mathcal{A}(\vec{\beta}(L_5) + \vec{\beta}(L_3)) = \mathcal{A}(1020\underline{2}0201) = 100100\underline{1}01001$, so, the result holds for $k = 2$. We induct on k. We assume that $\vec{\beta}(L_{2k-1} + L_{2k-3}) = 100100(10)^{k-3}\underline{1}(01)^{k-2}001$. Fact 4.2 implies that $\vec{\beta}(L_{2k} + L_{2k-2}) = 1010^{2k-3}\underline{0}0^{2k-3}101$. Therefore, $\vec{\beta}(L_{2k+1} + L_{2k-1}) = \vec{\beta}(L_{2k-1} + L_{2k} + L_{2k-3} + L_{2k-2}) = \mathcal{A}(\vec{\beta}(L_{2k-1} + L_{2k-3}) + \vec{\beta}(L_{2k} + L_{2k-2})) = \mathcal{A}(201100(10)^{k-3}\underline{1}(01)^{k-2}0111) = \mathcal{A}(201100(10)^{k-3}\underline{1}(01)^{k-2}01001)$. Note that this last vector is reduced to index $-2k+5$. The algorithm will not change any of the entries except that the 20110 that occurs on the left changes to 1001001. Thus, $\vec{\beta}(L_{2k+1} + L_{2k-1}) = 10010010(10)^{k-3}\underline{1}(01)^{k-2}01001$, and the induction is completed.

ACKNOWLEDGMENTS

Discussions with Laura Sanchis, Piero Filipponi, Dan Hendrick, and Ed Keppelmann were very helpful in the preparation of this paper.

REFERENCES

1. G. Bergman. "A Number System with an Irrational Base." *Math. Magazine* **31** (1957):98-110.
2. F. Blanchard. "β-Expansions and Symbolic Dynamics." *Theor. Comput. Sci.* **65** (1989): 131-41.
3. P. Filipponi. Personal correspondence, 1996.
4. P. Filipponi & H. T. Freitag. "The Zeckendorf Decomposition of Certain Classes of Integers." In *Applications of Fibonacci Numbers* **6**:123-35. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1996.
5. P. Filipponi & E. Hart. "The Zeckendorf Decomposition of Certain Fibonacci-Lucas Products." *The Fibonacci Quarterly* **36.3** (1998):240-47.
6. C. Frougny. "How To Write Integers in Non-Integer Base." In *LATIN 92, São Paulo*, pp. 154-64. Springer Lecture Notes in Computer Science, 1992.
7. C. Frougny. "Representations of Numbers and Finite Automata." *Math. Systems Theory* **25** (1992):37-60.

8. P. J. Grabner, I. Nemes, A. Pethö, & R. F. Tichy. "On the Least Significant Digit of Zeckendorf Expansions." *The Fibonacci Quarterly* **34.2** (1996):147-51.
9. P. J. Grabner, I. Nemes, A. Pethö, & R. F. Tichy. "Generalized Zeckendorf Decompositions." *Appl. Math. Letters* 7 (1994):25-28.
10. P. J. Grabner & R. F. Tichy. "Contributions to Digit Expansions with Respect to Linear Recurrences." *J. Number Theory* **36** (1990):160-69.
11. E. Hart & E. Keppelmann. "Explorations in Nielson Periodic Point Theory for the Double Torus." To appear in *Topology Appl.*
12. E. Hart & L. Sanchis. "The Occurrence of F_k in the Zeckendorf Decomposition of kF_k." To appear in *The Fibonacci Quarterly*.
13. C. G. Lekkerkerker. "Voorstelling van Natuurlijke Getallen door een Som van Getallen van Fibonacci." *Simon Stevin* **29** (1952):190-95.
14. W. Parry. "On the β-Expansions of Real Numbers." *Acta Math. Acad. Sci. Hung.* **8** (1960): 401-16.
15. A. Rényi. "Representations for Real Numbers and Their Ergodic Properties." *Acta Math. Acad. Sci. Hung.* **8** (1957):477-93.
16. S. Vajda. *Fibonacci and Lucas Numbers & the Golden Section*. Chichester: Ellis Horwood Ltd., 1989.
17. E. Zeckendorf. "Réprésentation des nombres naturels par une Somme de nombres de Fibonacci ou de nombres de Lucas." *Bull. Soc. Roy. Sci. Liège* **41** (1972):179-82.

AMS Classification Numbers: 11D85, 11A63, 11B39, 05A63

❖❖❖

NEW PROBLEM WEB SITE

Readers of *The Fibonacci Quarterly* will be pleased to know that many of its problems can now be searched electronically (at no charge) on the World Wide Web at

http://problems.math.umr.edu

Over 23,000 problems from 42 journals and 22 contests are references by the site, which was developed by Stanley Rabinowitz's MathPro Press. Ample hosting space for the site was generously provided by the Department of Mathematics and Statistics at the University of Missouri-Rolla, through Leon M. Hall, Chair.

Problem statements are included in most cases, along with proposers, solvers (whose solutions were published), and other relevant bibliographic information. Difficulty and subject matter vary widely; almost any mathematical topic can be found.

The site is being operated on a volunteer basis. Anyone who can donate journal issues or their time is encouraged to do so. For further information, write to

Mr. Mark Brown
Director of Operations, MathPro Press
1220 East West Highway #1010A
Silver Spring, MD 20910
(301) 587-0618 (Voice mail)
bowron@compuserve.com (e-mail)