POLYNOMIALS RELATED TO MORGAN-VOYCE POLYNOMIALS

Gospava B. Đorđević
University of Niš, Faculty of Technology, 16000 Leskovac, Yugoslavia
(Submitted April 1997-Final Revision March 1998)

1. INTRODUCTION

In this note we shall study two classes of polynomials, $\{P_{n,m}^{(r)}(x)\}$ and $\{Q_{n,m}^{(r)}(x)\}$, where r is integer. For $m=1$, these polynomials are the known polynomials $P_n^{(r)}(x)$ (see [1]) and $Q_n^{(r)}(x)$ (see [4]). Particularly, $P_n^{(r)}(x)$ and $Q_n^{(r)}(x)$ are the well-known classical Morgan-Voyce polynomials $b_n(x)$ and $B_n(x)$ (see [1], [2], [3], [4]). In Section 2 we shall study the class of polynomials $P_{n,m}^{(r)}(x)$. The polynomials $Q_{n,m}^{(r)}(x)$ are given in Section 3. The main results in this paper relate to the determination of coefficients of the polynomials $P_{n,m}^{(r)}(x)$ and $Q_{n,m}^{(r)}(x)$. Also, we give some interesting relations between the polynomials $P_{n,m}^{(r)}(x)$ and $Q_{n,m}^{(r)}(x)$.

2. POLYNOMIALS $P_{n,m}^{(r)}(x)$

We shall introduce the polynomials $P_{n,m}^{(r)}(x)$ by

$$P_{n,m}^{(r)}(x) = 2P_{n-1,m}^{(r)}(x) - P_{n-2,m}^{(r)}(x) + xP_{n-m,m}^{(r)}(x), \; n > m \tag{2.1}$$

with

$$P_{n,m}^{(r)}(x) = 1 + nr \text{ for } n = 0, 1, \ldots, m-1, \quad P_{m,m}^{(r)}(x) = 1 + mr + x. \tag{2.2}$$

So, by (2.1) and (2.2), we find the first $(m+2)$-members of the sequence $\{P_{n,m}^{(r)}(x)\}$:

$$\begin{aligned} &P_{0,m}^{(r)}(x) = 1, \quad P_{1,m}^{(r)}(x) = 1 + r, \ldots, P_{m,m}^{(r)}(x) = 1 + mr + x, \\ &P_{m+1,m}^{(r)}(x) = 1 + (m+1)r + (3+r)x. \end{aligned} \tag{2.3}$$

From (2.3), by induction on n, we see that there exists a sequence $\{b_{n,k}^{(r)}\}$ ($n \ge 0$ and $k \ge 0$) of numbers such that

$$P_{n,m}^{(r)}(x) = \sum_{k=0}^{[n/m]} b_{n,k}^{(r)} x^k, \tag{2.4}$$

with $b_{n,k}^{(r)} = 0$ for $k > [n/m]$.

By (2.4), we get

$$b_{n,0}^{(r)} = P_{n,m}^{(r)}(0). \tag{2.5}$$

Let us take $x=0$ in (2.1). Now, using (2.5), we obtain the following difference equation:

$$b_{n,0}^{(r)} = 2b_{n-1,0}^{(r)} - b_{n-2,0}^{(r)}, \; n \ge 2, m \ge 1, \tag{2.6}$$

with initial values $b_{0,0}^{(r)} = 1$ and $b_{1,0}^{(r)} = 1 + r$.

Solving (2.6), we get

$$b_{n,0}^{(r)} = 1 + nr, \; n \ge 0. \tag{2.7}$$

From (2.1), we obtain the following recurrence relation:

$$b_{n,k}^{(r)} = 2b_{n-1,k}^{(r)} - b_{n-2,k}^{(r)} + b_{n-m,k-1}^{(r)}, \; n \geq m, k \geq 1. \tag{2.8}$$

Next, we can write the sequence $\{b_{n,k}^{(r)}\}$ into the form of the general triangle:

TABLE 1

n/k	0	1	2	3	$\cdots$
1	1	$\cdots$	$\cdots$	$\cdots$	$\cdots$
2	$1+r$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$m-1$	$1+(m-1)r$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
m	$1+mr$	1	$\cdots$	$\cdots$	$\cdots$
$m+1$	$1+(m+1)r$	$3+r$	$\cdots$	$\cdots$	$\cdots$
$m+2$	$1+(m+2)r$	$6+4r$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Remark 1: For $m=1$, $r=0$ and $r=1$, Table 1 is exactly the *DFF* and the DFF_x triangle, respectively (see [2], [3]).

Theorem 2.1: The coefficients $b_{n,k}^{(r)}$ satisfy the relation

$$b_{n,k}^{(r)} = b_{n-1,k}^{(r)} + \sum_{s=0}^{n-m} b_{s,k-1}^{(r)}, \; n \geq m, k \geq 1. \tag{2.9}$$

Proof: We shall use induction on n. By direct computation, we see that (2.9) holds for every $n = 0, 1, \ldots, m-1$. If we suppose that (2.9) is true for n $(n \geq m)$, then, from (2.8) for $n+1$, we have

$$\begin{aligned} b_{n+1,k}^{(r)} &= 2b_{n,k}^{(r)} - b_{n-1,k}^{(r)} + b_{n+1-m,k-1}^{(r)} \\ &= b_{n,k}^{(r)} + b_{n-1,k}^{(r)} + \sum_{s=0}^{n-m} b_{s,k-1}^{(r)} + b_{n+1-m,k-1}^{(r)} - b_{n-1,k}^{(r)} \\ &= b_{n,k}^{(r)} + \sum_{s=0}^{n+1-m} b_{s,k-1}^{(r)}. \end{aligned}$$

Thus, statement (2.9) follows from the last equalities. □

One of the main results is given by the following theorem.

Theorem 2.2: For any $n \geq 0$ and any $k \geq 0$ such that $0 \leq k \leq [n/m]$, we get

$$b_{n,k}^{(r)} = \binom{n-(m-2)k}{2k} + r\binom{n-(m-2)k}{2k+1}, \tag{2.10}$$

where $\binom{p}{s} = 0$ for $s > p$.

Proof: We use induction on n. First, from (2.7), we see that (2.10) is true for $k=0$. Also, if $n=0,1,\dots,m-1$, then $k=0$, so (2.10) is true. Assume that (2.10) holds for $n-1$ $(n>m)$. Then, by (2.8) for n, we get

$$b_{n,k}^{(r)}=2b_{n-1,k}^{(r)}-b_{n-2,k}^{(r)}+b_{n-m,k-1}^{(r)}=x_{n,k}+ry_{n,k},$$

where

$$x_{n,k}=2\binom{n-1-(m-2)k}{2k}-\binom{n-2-(m-2)k}{2k}+\binom{n-m-(m-2)(k-1)}{2k-2}$$

and

$$y_{n,k}=2\binom{n-1-(m-2)k}{2k+1}-\binom{n-2-(m-2)k}{2k+1}+\binom{n-m-(m-2)(k-1)}{2k-1}.$$

Next, from the well-known relation

$$\binom{p}{s}=\binom{p-1}{s}+\binom{p-1}{s-1},$$

we find that

$$x_{n,k}=\binom{n-(m-2)k}{2k}\quad\text{and}\quad y_{n,k}=\binom{n-(m-2)k}{2k+1}.\ \square$$

Particular Cases

For $m=1$ and $r=0$, and for $m=1$ and $r=1$, by (2.10), we get

$$b_{n,k}^{(0)}=\binom{n+k}{2k}\quad\text{and}\quad b_{n,k}^{(1)}=\binom{n+k}{2k}+\binom{n+k}{2k+1}=\binom{n+1+k}{2k+1}.$$

These are the coefficients of the classical Morgan-Voyce polynomials $b_n(x)$ and $B_n(x)$, respectively (see [3], [4]). Namely, we have

$$b_{n+1}(x)=\sum_{k=0}^{n}\binom{n+k}{2k}x^k\quad\text{and}\quad B_{n+1}(x)=\sum_{k=0}^{n}\binom{n+1+k}{2k+1}x^k.$$

We shall now prove the following lemma.

Lemma 2.1:

$$b_{n,k}^{(1)}-b_{n-2,k}^{(1)}=b_{n,k}^{(0)}+b_{n-1,k}^{(0)},\ n\geq 2. \tag{2.11}$$

Proof: From (2.10), for $r=1$, we get

$$b_{n,k}^{(1)}-b_{n-2,k}^{(1)}=\binom{n-(m-2)k}{2k}+\binom{n-(m-2)k}{2k+1}-\binom{n-2-(m-2)k}{2k}-\binom{n-2-(m-2)k}{2k+1}$$

$$=\binom{n-(m-2)k}{2k}+\binom{n-1-(m-2)k}{2k}=b_{n,k}^{(0)}+b_{n-1,k}^{(0)}.$$

From the last equalities, we get (2.11). $\square$

Remark 2: For $m = 1$, from (2.11), we obtain (see [5])

$$B_n(x) - B_{n-2}(x) = b_n(x) + b_{n-1}(x),$$

where $B_n(x)$ and $b_n(x)$ are the classical Morgan-Voyce polynomials.

3. POLYNOMIALS $Q_{n,m}^{(r)}(x)$

First, we are going to define the polynomials $Q_{n,m}^{(r)}(x)$, which are the generalization of the polynomials $Q_n^{(r)}(x)$ (see [4]). The polynomials $Q_{n,m}^{(r)}(x)$ are given by

$$Q_{n,m}^{(r)}(x) = 2Q_{n-1,m}^{(r)}(x) - Q_{n-2,m}^{(r)}(x) + xQ_{n-m,m}^{(r)}(x), \quad n \geq m, \tag{3.1}$$

with the initial values

$$Q_{n,m}^{(r)}(x) = 2 + nr \text{ for } n = 0, 1, \ldots, m-1, \quad Q_{m,m}^{(r)}(x) = 2 + mr + x. \tag{3.2}$$

From (3.2) and (3.1), by induction on n, we see that there exists a sequence $\{d_{n,k}^{(r)}\}$ ($n \geq 0$ and $k \geq 0$) of integers such that

$$Q_{n,m}^{(r)}(x) = \sum_{k=0}^{[n/m]} d_{n,k}^{(r)} x^k, \tag{3.3}$$

where

$$d_{n,n}^{(r)} = \begin{cases} 1, & n \geq 1, \\ 2, & n = 0. \end{cases} \tag{3.4}$$

From (3.3), we get

$$Q_{n,m}^{(r)}(0) = d_{n,0}^{(r)}.$$

Thus, by (3.1) and (3.2), we have

$$d_{n,0}^{(r)} = 2d_{n-1,0}^{(r)} - d_{n-2,0}^{(r)} \quad (n \geq 2), \tag{3.5}$$

with

$$d_{0,0}^{(r)} = 2 \quad \text{and} \quad d_{1,0}^{(r)} = 2 + r. \tag{3.6}$$

Solving (3.5), by (3.6), we obtain

$$d_{n,0}^{(r)} = 2 + nr, \quad n \geq 0. \tag{3.7}$$

Furthermore, from (3.1), we get

$$d_{n,k}^{(r)} = 2d_{n-1,k}^{(r)} - d_{n-2,k}^{(r)} + d_{n-m,k-1}^{(r)} \quad (n \geq m, m \geq 1, k \geq 1). \tag{3.8}$$

In Table 2, we write the coefficients $d_{n,k}^{(r)}$. Thus, from Tables 1 and 2, we see that

$$d_{n,k}^{(r)} = b_{n,k}^{(r)} + b_{n-1,k}^{(0)}, \quad n = 0, 1, \ldots, m-1.$$

TABLE 2

n/k	0	1	2	$\cdots$
0	2	$\cdots$	$\cdots$	$\cdots$
1	$2+r$	$\cdots$	$\cdots$	$\cdots$
2	$2+r$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$m-1$	$2+(m-1)r$	$\cdots$	$\cdots$	$\cdots$
m	$2+mr$	1	$\cdots$	$\cdots$
$m+1$	$2+(m+1)r$	$4+r$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Now we shall prove the following theorem.

Theorem 3.1: For $n \geq 1$, the following equalities hold:

$$\begin{aligned} d_{n,k}^{(r)} &= b_{n,k}^{(r)} + b_{n-1,k}^{(0)} \\ &= \binom{n-(m-2)k}{2k} + \binom{n-1-(m-2)k}{2k} + r\binom{n-(m-2)k}{2k+1}. \end{aligned} \tag{3.9}$$

Proof: In the proof, we use induction on n. For $n=1$, by direct computation, we conclude that (3.9) is true. We assume that (3.9) is true for n $(n \geq 1)$. Then, for $n+1$, we get

$$\begin{aligned} b_{n+1,k}^{(r)} + b_{n,k}^{(0)} &= 2b_{n,k}^{(r)} - b_{n-1,k}^{(r)} + b_{n+1-m,k-1}^{(r)} + 2b_{n-1,k}^{(0)} - b_{n-2,k}^{(0)} + b_{n-m,k-1}^{(0)} \quad \text{[by (2.8)]} \\ &= 2\left(b_{n,k}^{(r)} + b_{n-1,k}^{(0)}\right) - \left(b_{n-1,k}^{(r)} + b_{n-2,k}^{(0)}\right) + b_{n+1-m,k-1}^{(r)} + b_{n-m,k-1}^{(0)} \\ &= 2d_{n,k}^{(r)} - d_{n-1,k}^{(r)} + d_{n+1-m,k-1}^{(r)} = d_{n+1,k}^{(r)} \quad \text{[by (3.8)]}. \end{aligned}$$

Now, from (2.10), we obtain (3.9). This completes the proof. □

Corollary 1:

$$d_{n,k}^{(r)} = \frac{n-(m-1)k}{k}\binom{n-1-(m-2)k}{2k-1} + r\binom{n-(m-2)k}{2k+1}.$$

Hence, for $m=1$ and $k>0$, we get (see [4])

$$d_{n,k}^{(r)} = \frac{n}{k}\binom{n-1+k}{2k-1} + r\binom{n+k}{2k+1}.$$

Corollary 2:

$$Q_{n,1}^{(r)}(1) = L_{2n} + rF_{2n} \quad \text{(see [4])}.$$

Corollary 3:

$$Q_{n,1}^{(2u+1)}(1) = 2P_{n,1}^{(u)} \quad \text{(see [4])}.$$

Theorem 3.2: The polynomials $P_{n,m}^{(r)}(x)$ and $Q_{n,m}^{(r)}(x)$ satisfy the relation

$$Q_{n,m}^{(r)}(x) = P_{n,m}^{(r)}(x) + P_{n-1,m}^{(0)}(x), \quad n \geq 1. \tag{3.10}$$

Proof: Multiply both sides of (3.9) by x^k and sum. Immediately, from (2.4) and (3.3), we obtain (3.10). □

Remark 3: For $m = 1$, (3.10) becomes (see [4])

$$Q_n^{(r)}(x) = P_n^{(r)}(x) + P_{n-1}^{(0)}(x), \quad n \geq 1.$$

Theorem 3.3:

$$Q_{n,m}^{(0)}(x) = P_{n,m}^{(1)}(x) - P_{n-2,m}^{(1)}(x).$$

Proof:

$$\begin{aligned} Q_{n,m}^{(0)}(x) &= \sum_{k=0}^{[n/m]} d_{n,k}^{(0)} x^k && \text{[by (3.3)]} \\ &= \sum_{k=0}^{[n/m]} (b_{n,k}^{(0)} + b_{n-1,k}^{(0)}) x^k && \text{[by (3.9)]} \\ &= \sum_{k=0}^{[n/m]} (b_{n,k}^{(1)} + b_{n-2,k}^{(1)}) x^k && \text{[by (2.11)]} \\ &= P_{n,m}^{(1)}(x) - P_{n-2,m}^{(1)}(x) && \text{[by (2.4)]}. \ \square \end{aligned}$$

Corollary 4: For $m = 1$, we get (see [4])

$$Q_n^{(0)}(x) = P_n^{(1)}(x) - P_{n-2}^{(1)}(x) = B_{n+1}(x) - B_{n-1}(x).$$

Thus, we obtain

$$Q_n^{(0)}(x) = \sum_{k=1}^{n} \frac{n}{k} \binom{n-1+k}{2k-1} x^k + 2.$$

REFERENCES

1. R. André-Jeannin. "A Generalization of Morgan-Voyce Polynomials." *The Fibonacci Quarterly* **32.3** (1994):228-31.
2. R. André-Jeannin. "A Note on a General Class of Polynomials, Part II." *The Fibonacci Quarterly* **33.4** (1995):341-51.
3. J. Lahr. "Fibonacci and Lucas Numbers and the Morgan-Voyce Polynomials in Ladder Networks and in Electrical Line Theory." In *Fibonacci Numbers and Their Applications* **1:**141-146. Ed. G. E. Bergum, A. N. Philippou, & A. F. Horadam. Dordrecht: Kluwer, 1986.
4. A. F. Horadam. "Polynomials Associated with Generalized Morgan-Voyce Polynomials." *The Fibonacci Quarterly* **34.4** (1996):342-49.
5. M. N. S. Swamy. "Properties of the Polynomials Defined by Morgan-Voyce." *The Fibonacci Quarterly* **4.1** (1996):73-81.

AMS Classification Numbers: 11B39, 26A24, 11B83

❖❖❖