FIBONACCI MATRICES

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1. INTRODUCTION

In this paper we discuss the Fibonacci matrices which are matrices whose elements are the classical Fibonacci numbers. Some properties are given for these matrices. The relations between these matrices and the units of the field $\mathcal{Q}(\theta)$, $(\theta^4+\theta^3+\theta^2+\theta+1=0)$ is also discussed. As an application, we deduce an interesting relation which includes the Fibonacci and Lucas numbers by using the properties of these Fibonacci matrices.

2. FIBONACCI NUMBERS AND FIBONACCI MATRICES

It is well known that if $\mathcal{Q}=\begin{pmatrix}11\\10\end{pmatrix}$ then $\mathcal{Q}^n=\begin{pmatrix}F_{n+1} & F_n\\ F_n & F_{n-1}\end{pmatrix}$. Many Fibonacci and Lucas identities have been developed using $\mathcal{Q}$ (see [1]).

We are interested in finding other matrices like $\mathcal{Q}$ so that the n^{th} power of the matrix has only the Fibonacci numbers as its elements. If such matrices exist, then we want to know their properties and what relations exist between the matrices and the Fibonacci numbers.

For matrices of order 2, we examine the set

$$F=\left\{\begin{pmatrix}\ell_1 & \ell_2\\ \ell_3 & \ell_4\end{pmatrix} \middle| \ell_i=0 \text{ or } 1\right\}.$$

One can easily see that the only matrices that work are

$$\mathcal{Q}_1=\begin{pmatrix}01\\11\end{pmatrix},\ \mathcal{Q}_2=\begin{pmatrix}11\\10\end{pmatrix},\ \pm\mathcal{Q}_1,\ \pm\mathcal{Q}_2,\ \pm\mathcal{Q}_1^{-1},\ \pm\mathcal{Q}_2^{-1}.$$

For matrices of order 4, we let

$$H_1=\begin{pmatrix}0&1&0&0\\1&1&0&0\\0&0&0&1\\0&0&1&1\end{pmatrix},\quad H_2=\begin{pmatrix}0&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&0\end{pmatrix}.$$

Note that H_1 and H_2 behave like $\mathcal{Q}$ and are made up of submatrices that are Fibonacci matrices of order 2, the null matrix or matrices that have properties similar to the Fibonacci matrices of order 2. However, H_1 and H_2 are not irreducible, so we ask whether there exists any irreducible matrix of order 4 that behaves like $\mathcal{Q}$.

Definition: A square matrix $\mathcal{A}$ of order r with integer elements is called a Fibonacci matrix if and only if:

(a) $\mathcal{A}^n$, $n=1,2,\ldots$ has only Fibonacci numbers as its elements. The elements may be positive, zero, or negative.

(b) $\mathcal{A}^n$ is irreducible (a matrix B is called irreducible if the matrix B cannot be reduced to a block diagonal matrix by permuting some rows or some columns). Our definition of irreducible is different from the common definition in order to avoid combining two Fibonacci matrices of order 2 to obtain a Fibonacci matrix of order 4.

(c) $\{\mathscr{A}^n \mid n \geq 1\}$ has $F_i \in \mathscr{A}^n$ for all i and some n.

$\mathscr{A}$ is called a basic Fibonacci matrix if $\mathscr{A}$ has only 1, −1, and 0 as its elements.

3. FIBONACCI MATRICES OF ORDER 4 AND THEIR PROPERTIES

Proposition 1: The matrices

$$\begin{pmatrix} -F_1 & -F_2 & -F_1 & 0 \\ 0 & -F_1 & -F_2 & -F_1 \\ F_1 & F_1 & 0 & -F_0 \\ F_0 & F_2 & F_2 & F_0 \end{pmatrix}, \begin{pmatrix} -F_1 & -F_1 & 0 & -F_2 \\ F_2 & F_0 & F_0 & F_2 \\ -F_2 & 0 & -F_1 & -F_1 \\ F_1 & -F_0 & F_1 & 0 \end{pmatrix}, \begin{pmatrix} -F_1 & -F_1 & 0 & F_0 \\ -F_0 & -F_2 & -F_2 & -F_0 \\ F_0 & 0 & -F_1 & -F_1 \\ F_1 & F_2 & F_1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} -F_2 & 0 & -F_1 & -F_1 \\ F_1 & -F_0 & F_1 & 0 \\ 0 & F_1 & -F_0 & F_1 \\ -F_1 & -F_1 & 0 & -F_2 \end{pmatrix}, \begin{pmatrix} -F_1 & F_0 & -F_1 & 0 \\ 0 & -F_1 & F_0 & -F_1 \\ F_1 & F_1 & 0 & F_2 \\ -F_2 & -F_0 & -F_0 & -F_2 \end{pmatrix}, \begin{pmatrix} -F_2 & -F_0 & -F_0 & -F_2 \\ F_2 & 0 & F_1 & F_1 \\ -F_1 & F_0 & -F_1 & 0 \\ 0 & -F_1 & F_0 & -F_1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -F_1 & -F_2 & -F_1 \\ F_1 & F_1 & 0 & -F_0 \\ F_0 & F_2 & F_2 & F_0 \\ -F_0 & 0 & F_1 & F_1 \end{pmatrix}, \begin{pmatrix} -F_0 & -F_2 & -F_2 & -F_0 \\ F_0 & 0 & -F_1 & -F_1 \\ F_1 & F_2 & F_1 & 0 \\ 0 & F_1 & F_2 & F_1 \end{pmatrix}, \begin{pmatrix} 0 & -F_1 & F_0 & -F_1 \\ F_1 & F_1 & 0 & F_2 \\ -F_2 & -F_0 & -F_0 & -F_2 \\ F_2 & 0 & F_1 & F_1 \end{pmatrix},$$

and

$$\begin{pmatrix} F_0 & 0 & -F_1 & -F_1 \\ F_1 & F_2 & F_1 & 0 \\ 0 & F_1 & F_2 & F_1 \\ -F_1 & -F_1 & 0 & F_0 \end{pmatrix}$$

are all basic Fibonacci matrices. We denote these matrices, respectively, by $\overline{F}_1, \overline{F}_2, \ldots, \overline{F}_{10}$.

Proof: For $\overline{F}_1$, we can easily calculate $\overline{F}_1^2, \overline{F}_1^3, \ldots, \overline{F}_1^{10}$. For example,

$$\overline{F}_1^{10} = \begin{pmatrix} F_9 & 0 & -F_{10} & -F_{10} \\ F_{10} & F_{11} & F_{10} & 0 \\ 0 & F_{10} & F_{11} & F_{10} \\ -F_{10} & -F_{10} & 0 & F_9 \end{pmatrix}.$$

By using the basic definition and the well-known properties of the Fibonacci numbers, one can easily prove by induction that

$$\overline{F}_1^{10k} = \begin{pmatrix} F_{10k-1} & 0 & -F_{10k} & -F_{10k} \\ F_{10k} & F_{10k+1} & F_{10k} & 0 \\ 0 & F_{10k} & F_{10k+1} & F_{10k} \\ -F_{10k} & -F_{10k} & 0 & F_{10k-1} \end{pmatrix}, \quad k = 1, 2, \ldots.$$

If we multiply $\overline{F}_1^i$, $i = 1, 2, \ldots, 9$, by $\overline{F}_1^{10k}$, $k = 1, 2, \ldots$, and use the basic properties of the Fibonacci numbers, we have the 10 patterns $\overline{F}_1^{10k+i}$. For example,

$$\overline{F}_1^{10k+1} = \begin{pmatrix} -F_{10k+1} & -F_{10k+2} & -F_{10k+1} & 0 \\ 0 & -F_{10k+1} & -F_{10k+2} & -F_{10k+1} \\ F_{10k+1} & F_{10k+1} & 0 & -F_{10k} \\ F_{10k} & F_{10k+2} & F_{10k+2} & F_{10k} \end{pmatrix},$$

$$\overline{F}_1^{10k+2} = \begin{pmatrix} 0 & F_{10k+2} & F_{10k+3} & F_{10k+2} \\ -F_{10k+1} & -F_{10k+2} & 0 & F_{10k+1} \\ -F_{10k+1} & -F_{10k+3} & -F_{10k+3} & -F_{10k+1} \\ F_{10k+1} & 0 & -F_{10k+2} & -F_{10k+2} \end{pmatrix}.$$

This completes the proof of part (a) of the Definition for $\overline{F}_1$. Part (b) of the Definition can be proved by the exhaustive method for all permutations of rows and columns. Part (c) of the Definition is obvious. Similar proof exists for $\overline{F}_2, \ldots, \overline{F}_{10}$. We would like to observe that the proofs for $\overline{F}_4$ and $\overline{F}_{10}$ can be simpler.

Proposition 2: If $\overline{F}_k$ is a Fibonacci matrix, then so is $-\overline{F}_k$, $k = 1, \ldots, 10$.

Proof: This is obvious since $(-\overline{F})^n = \pm\overline{F}^n$, $n = 1, 2, \ldots$.

We now let $F_{21-k} = -F_k$, $k = 1, \ldots, 10$.

Proposition 3: If $\overline{F}_k$ is a Fibonacci matrix, then so is $\overline{F}_k^T$, $k = 1, \ldots, 20$, where $\mathscr{A}^T$ denotes the transpose of the matrix $\mathscr{A}$.

Proof: This is obvious since $(\overline{F}_k^T)^n = (\overline{F}_k^n)^T$.

Thus, we obtain 40 Fibonacci matrices of order 4. However, it is sufficient to discuss only $\overline{F}_1, \ldots, \overline{F}_{20}$. We let $\mathfrak{F} = \{\overline{F}_k \mid k = 1, \ldots, 20\}$.

Proposition 4: If $F_k \in \mathfrak{F}$, then $\overline{F}_k^{-1} \in \mathfrak{F}$ for $k = 1, \ldots, 20$.

Proof: It is not difficult to verify that $\overline{F}_1^{-1} = \overline{F}_6$, $\overline{F}_2^{-1} = \overline{F}_{14}$, $\overline{F}_3^{-1} = \overline{F}_{12}$, $\overline{F}_4^{-1} = \overline{F}_{10}$, and $\overline{F}_5^{-1} = \overline{F}_{13}$. The rest can be proved by using the relations $\overline{F}_{21-k} = -\overline{F}_k$, $k = 1, \ldots, 10$. Another interesting result is the following.

Proposition 5: If $\overline{F}_k \in \mathfrak{F}$, then $\det(\overline{F}_k) = 1$, where the $\det(A)$ denotes the determinant of matrix A.

It is well known that, in general, the multiplication of matrices is noncommutative. However, for these Fibonacci matrices, we have the following.

Proposition 6: If $\overline{F}_k$, $\overline{F}_h \in \mathfrak{F}$, then $\overline{F}_k\overline{F}_h = \overline{F}_h\overline{F}_k$.

Proof: One can easily verify that this is true. In order to investigate the properties of multiplication for the matrices in $\mathfrak{F}$, we start by studying the following 10 matrices. Let

$$A_1 = A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad A_i = A^i,\ i = 1, 2, \ldots.$$

It is easy to verify that $A_5 = -E$ and $A_{10} = E$, where the E is the identify a matrix of order 4. Let $\mathscr{A} = \{A_k \mid k = 1, 2, \ldots, 10\}$. Obviously, the multiplication group of $\mathscr{A}$ is isomorphic to the group $(\gamma^k \mid \gamma = \exp(2\pi i/10),\ k = 0, 1, 2, \ldots, 9)$. It is also easy to verify that $A_k\overline{F}_h = \overline{F}_h A_k$ or that the multiplication of the A's and $\overline{F}$'s matrices is commutative. Furthermore, one can easily show that $\mathscr{A}\mathfrak{F} \subset \mathfrak{F}$. In fact, we have the following multiplication table, where the product array is $\overline{F}_n = A_k\overline{F}_h$. For example, $\overline{F}_{19} = A_4\overline{F}_9$. From the table and the properties of A_k, it is easy to see the results in Proposition 7.

$k \backslash h$	1	2	3	4	5	6	7	8	9	10
1	10	9	7	6	4	2	20	18	16	13
2	13	16	20	2	6	9	11	14	17	3
3	3	17	11	9	2	16	8	1	15	7
4	7	15	8	16	9	17	18	10	19	20
5	20	19	18	17	16	15	14	13	12	11
6	11	12	14	15	17	19	1	3	5	8
7	8	5	1	19	15	12	10	7	4	18
8	18	4	10	12	19	5	13	20	6	14
9	14	6	13	5	12	4	3	11	2	1
10	1	2	3	4	5	6	7	8	9	10

Proposition 7: Let $\mathfrak{F}_1 = \{\overline{F}_k \mid k = 1, 3, 7, 8, 10, 11, 13, 14, 18, 20\}$ and $\mathfrak{F}_2 = \mathfrak{F} \setminus \mathfrak{F}_1$. Then

- ***(a)*** For $\overline{F}_k\overline{F}_h \in \mathfrak{F}_i$, there exists $\overline{F}_n \in \mathfrak{F}_i$ such that $\overline{F}_k\overline{F}_h = \pm\overline{F}_n^2$, $i = 1, 2$.
- ***(b)*** For $\overline{F}_k \in \mathfrak{F}_1$, $\overline{F}_h \in \mathfrak{F}_2$, there exist $A_n \in \mathscr{A}$ such that $\overline{F}_k\overline{F}_h = A_n$.
- ***(c)*** For any $\overline{F}_k, \overline{F}_h \in \mathfrak{F}_i$, $\overline{F}_k^{10n} = \overline{F}_h^{10n}$, where $n = 0, 1, 2, \ldots,$ and $i = 1, 2$.

The proof is omitted since it is very straightforward.

4. THE CHARACTERISTIC POLYNOMIAL, CHARACTERISTIC VALUE, AND CHARACTERISTIC VECTOR OF A FIBONACCI MATRIX

It is not difficult to compute the characteristic polynomial for $\overline{F}_k \in \mathfrak{F}$.

Proposition 8: The characteristic polynomials of $\overline{F}_1$ (or $\overline{F}_2$), $\overline{F}_3$ (or $\overline{F}_5$), and $\overline{F}_4$ are, respectively, $\lambda^4 + 2\lambda^3 + 4\lambda^2 + 3\lambda + 1$, $\lambda^4 + 3\lambda^3 + 4\lambda^2 + 2\lambda + 1$, and $\lambda^4 + 2\lambda^3 - \lambda^2 - 2\lambda + 1$. The other characteristic polynomials can easily be reduced by using Proposition 4 and the fact that $\overline{F}_{21-k} = -\overline{F}_k$. The proofs are omitted.

There is a very nice property for $\overline{F}_k$ if $k \equiv h \pmod{10}$. We can verify the following.

Proposition 9: Let $G_k = \overline{F}_k^{10}$, $k = 1, \ldots, 20$. Then:

(a) G_k only takes one of two forms, i.e., either

$$G_k = G_1 = \begin{pmatrix} F_9 & 0 & -F_{10} & -F_{10} \\ F_{10} & F_{11} & F_{10} & 0 \\ 0 & F_{10} & F_{11} & F_{10} \\ -F_{10} & -F_{10} & 0 & F_9 \end{pmatrix} \text{ or } G_k = G_2 = \begin{pmatrix} F_{11} & 0 & F_{10} & F_{10} \\ -F_{10} & F_9 & -F_{10} & 0 \\ 0 & -F_{10} & F_9 & -F_{10} \\ F_{10} & F_{10} & 0 & F_{11} \end{pmatrix};$$

(b) $G_1 = G_2^{-1}$;

(c) G_k, $k = 1, \ldots, 20$, all satisfy the same characteristic equation,

$$G_k^4 - 246G_k^3 + 15131G_k^2 - 246G_k + E = 0.$$

We conclude this section by giving some properties of the characteristic roots of the Fibonacci matrices and looking at the characteristic vectors of $\overline{F}_n$.

Theorem 1:

(A) Each characteristic root of $\overline{F}_h \in \mathfrak{F}$ is a linear combination of $\exp(2\pi ik/5)$, $k = 0, 1, 2, 3$, with integer coefficients.

(B) Each characteristic root of $\overline{F}_k^n$, $n = 1, 2, \ldots$, is a linear combination of $\exp(2\pi ik/5)$, $k = 0, 1, 2, 3$, with integer coefficients.

Proof:

(A) One method of proof is the following. Let the elements of the first row of $\overline{F}_h$ be f_{h11}, f_{h12}, f_{h13}, f_{h14}, $h = 1, \ldots, 20$, and let $\theta_k = \exp(2\pi ik/5)$, $k = 0, 1, 2, 3, 4$. It is easy to verify that the $\sum_{j=1}^4 f_{h1j}\theta_k^{j-1}$ are the roots of the characteristic equation of $\overline{F}_h$. Noticing that $\sum_{k=0}^4 \theta_k = 0$, we see that θ_k, $k = 4, 5, \ldots$, can be written as a linear combination of θ_k, $k = 0, 1, 2, 3$, with integer coefficients. Hence, the conclusion of (A) is true.

(B) We notice that $|\lambda E - A| = 0$ implies that $|\lambda^n E - A^n| = |\lambda E - A| \cdot |\lambda^{n-1}E + \lambda^{n-2}A + \cdots + A^{n-1}| = 0$. Hence, it follows that the characteristic root of $\overline{F}_h^n$ is λ^n, where λ is the characteristic root of $\overline{F}_h$. Looking at the proof of (A), the proof of (B) is now obvious.

Concerning the characteristic vector of $\overline{F}_k^n$, we have the following theorem.

Theorem 2: Let $\theta_k = (1, \theta_k, \theta_k^2, \theta_k^3)^T$, $\theta_k = \exp(2\pi i/5)$, $k = 1, 2, 3, 4$, and let the f_{h1j}'s have the same meaning as in the proof of Theorem 1(A). Then:

(A) θ_k is the characteristic vector of $\overline{F}_h$ corresponding to the characteristic value of $\sum_{j=1}^4 f_{h1j}\theta_k^{j-1}$, $k = 1, 2, 3, 4$;

(B) θ_k is the characteristic vector of $\overline{F}_h^n$ corresponding to the characteristic value of $(\sum_{j=1}^4 f_{h1j}\theta_k^{j-1})^n$, $k = 1, 2, 3, 4$, $n = 1, 2, \ldots$.

Proof:

(A) The proof of this is trivial.

(B) First, we notice that $(\lambda E - A) = 0$ implies

$$\lambda^n E - A^n = (\lambda^{n-1}E + \lambda^{n-2}A + \cdots + A^{n-1})(\lambda E - A) = 0.$$

The conclusion is now obtained directly from (A).

5. APPLICATIONS AND REMARKS

When we proved Proposition 1, we saw that the proofs for $\overline{F}_4$ and $\overline{F}_{10}$ could be simpler. That is so because the patterns of the signs for their powers has relatively small numbers. The matrix ($\operatorname{sgn} a_{ij}$) is called the pattern of signs for the matrix (a_{ij}), where we have

$$\operatorname{sgn} x = \begin{cases} 1, & x > 0, \\ 0, & x = 0 \\ -1, & x < 0. \end{cases}$$

One can easily verify that the pattern of signs for $\overline{F}_h^n, n = 1, 2, \ldots,$ is a periodic function of n that the period is never more than ten. In fact, one can easily compute the following:

$$\text{The period of sign's pattern for } \overline{F}_k = \begin{cases} 10, & \text{when } k = 1,2,7,12,13,15,16,18, \\ 5, & \text{when } k = 3,5,6,8,9,14,19,20, \\ 2, & \text{when } k = 4,11, \\ 1, & \text{when } k = 10,17. \end{cases}$$

It is worth mentioning that in the sign's pattern of $\overline{F}_k$, even $F_0 = 0$, we understand that the F_0 has a positive or negative sign.

As an application, we look at $\overline{F}_{17}$ and deduce some wonderful relations between the Fibonacci and Lucas numbers.

By a tedious and careful investigation, one can obtain many relations like the following.

Theorem 3: For n odd, we have

$$F_{4n+2} - 2L_nF_{3n+2} + (L_n^2 - 2)F_{2n+2} + 2L_nF_{n+2} + 1 = 0, \tag{1}$$
$$F_{4n+1} - 2L_nF_{3n+1} + (L_n^2 - 2)F_{2n+1} + 2L_nF_{n+1} + 1 = 0, \tag{2}$$
$$F_{4n-1} - 2L_nF_{3n-1} + (L_n^2 - 2)F_{2n-1} + 2L_nF_{n-1} + 1 = 0, \tag{3}$$
$$F_{4n} - 2L_nF_{3n} + (L_n^2 - 2)F_{2n} + 2L_nF_n = 0. \tag{4}$$

For n even, we have

$$F_{4n+2} - 2L_nF_{3n+2} + (L_n^2 + 2)F_{2n+2} - 2L_nF_{n+2} + 1 = 0, \tag{5}$$
$$F_{4n+1} - 2L_nF_{3n+1} + (L_n^2 + 2)F_{2n+1} - 2L_nF_{n+1} + 1 = 0, \tag{6}$$
$$F_{4n-1} - 2L_nF_{3n-1} + (L_n^2 + 2)F_{2n-1} - 2L_nF_{n-1} + 1 = 0, \tag{7}$$
$$F_{4n} - 2L_nF_{3n} + (L_n^2 + 2)F_{2n} - 2L_nF_n = 0. \tag{8}$$

In order to prove Theorem 3, we need the following proposition.

Proposition 10: Let S_n denote the sum of all principal 2 minors of $\overline{F}_{17}^n$. Then

$$S_n = \begin{cases} L_n^2 + 2 & \text{when } n \text{ is even,} \\ L_n^2 - 2 & \text{when } n \text{ is odd.} \end{cases}$$

A careful examination of $\overline{F}_{17}$ will show that Proposition 10 is equivalent to the following.

Proposition 11: $F_{n-1}^2 + 4F_{n-1}F_{n+1} + F_{n+1}^2 - 2F_n^2 = \begin{cases} L_n^2 + 2 & \text{when } n \text{ is even,} \\ L_n^2 - 2 & \text{when } n \text{ is odd.} \end{cases}$

Proof: Obviously, the left side of this equation is equal to $(F_{n-1} + F_{n+1})^2 + 2(F_{n-1}F_{n+1} - F_n^2)$. However, this is equal to the right side since we have $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$ and $L_n = F_{n-1} + F_{n+1}$.

We now give the proof of Theorem 3. Using the relation between the coefficients of the characteristic polynomial and the principal minors of a matrix, applying Proposition 10, and doing some proper computation, we can see that the characteristic equation for the matrix $G_n = \overline{F}_{17}^n$ is $\lambda_n^4 - 2L_n\lambda_n^3 + (L_n^2 - 2)\lambda_n^2 + 2L_n\lambda_n + 1 = 0$ when n is odd. Hence, we have

$$\overline{F}_{17}^{4n} - 2L_n\overline{F}_{17}^{3n} + (L_n^2 - 2)\overline{F}_{17}^{2n} + 2L_n\overline{F}_{17}^n + E = 0$$

by the Hamilton-Cayley theorem. Substituting $\overline{F}_{17}$'s expression by F_n into the last equality and comparing the coefficients of the (1, 1)-, (2, 2)-, and (2, 3)-elements of the resulting matrices, we obtain (2), (3), and (4) and $(1) = (2) + (4)$.

In a similar manner, we can prove the results when n is even.

Remark 1: One can find a more uniform pattern than is given in Theorem 3 using the following proposition as an example.

Proposition 12: The sum of all principal 2-minors of $\overline{F}_{18}^n$ is equal to $L_n^2 \pm 3$.

Remark 2: In this paper, it is shown that the Fibonacci matrices play an important role in the connection between the ancient Fibonacci numbers and some properties of the field $\mathfrak{Q}(\theta)$, where the θ is a zero of the polynomial $x^4 + x^3 + x^2 + x + 1 = 0$.

Remark 3: Research problems.

(a) Are there other Fibonacci matrices of order 4 besides the 40 matrices dealt with in this paper?

(b) Are there any Fibonacci matrices of order higher than 4?

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