APPLICATION OF MARKOV CHAINS PROPERTIES TO r-GENERALIZED FIBONACCI SEQUENCES

Mehdi Mouline and Mustapha Rachidi

Département de Mathématiques et Informatique

Faculté des Sciences, Université Mohammed V, P.B. 1014, Rabat, Morocco

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1. INTRODUCTION

Let $a_0, \ldots, a_{r-1}$ $(r \geq 2, a_{r-1} \neq 0)$ be some fixed real numbers. An r-generalized Fibonacci sequence $\{V_n\}_{n=0}^{+\infty}$ is defined by the linear recurrence relation of order r,

$$V_{n+1} = a_0 V_n + a_1 V_{n-1} + \cdots + a_{r-1} V_{n-r+1}, \text{ for } n \geq r-1, \tag{1}$$

where $V_0, \ldots, V_{r-1}$ are specified by the initial conditions. A first connection between Markov chains and sequence (1), whose coefficients a_i $(0 \leq i \leq r-1)$ are nonnegative, is considered in [6]. And we established that the limit of the ratio V_n / q^n exists if and only if $\mathrm{CGD}\{i+1; a_i > 0\} = 1$, where CGD means the common great divisor and q is the unique positive root of the characteristic polynomial $P(x) = x^r - a_0 x^{r-1} - \cdots - a_{r-2} x - a_{r-1}$ (cf. [6] and [7]).

Our purpose in this paper is to give a second connection between Markov chains and sequence (1) when the a_i are nonnegative. This allows us to express the general term V_n $(n \geq r)$ in a combinatoric form. Note that the combinatoric form of V_n has been studied by various methods and techniques (cf. [2], [4], [5], [8], [9], and [10], for example). However, our method is different from those above, and it allows us to study the asymptotic behavior of the ratio V_n / q^n, from which we derive a new approximation of the number q.

This paper is organized as follows. In Section 2 we study the connection between Markov chains and sequence (1) when the coefficients a_j are nonnegative and $a_0 + \cdots + a_{r-1} = 1$, and we establish the combinatoric form of V_n for $n \geq r$. In Section 3 we are interested in the asymptotic behavior of V_n when the coefficients a_j are arbitrary nonnegative real numbers.

2. COMBINATORIC FORM OF SEQUENCE (1) WITH NONNEGATIVE COEFFICIENTS OF SUM 1

2.1 Sequence (1) and Markov Chains

Let $\{V_n\}_{n=0}^{+\infty}$ be a sequence (1) whose coefficients $a_0, \ldots, a_{r-1}$ $(a_{r-1} \neq 0)$ are nonnegative with $a_0 + \cdots + a_{r-1} = 1$. Set

$$X = \begin{pmatrix} V_0 \\ V_1 \\ \vdots \\ V_n \\ \vdots \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} & I_r & & \mid & & 0 & & & \\ a_{r-1} & a_{r-2} & \cdots & a_0 & 0 & \cdots & & & \\ 0 & a_{r-1} & a_{r-2} & \cdots & a_0 & 0 & \cdots & & \\ 0 & 0 & a_{r-1} & a_{r-2} & \cdots & a_0 & 0 & \cdots & \\ \vdots & & & & & & & & \end{pmatrix}, \tag{2}$$

where I_r is the identity $r \times r$ matrix. The condition $\sum_{i=0}^{r-1} a_i = 1$ implies that $P = (P(n,m))_{n \geq 0, m \geq 0}$ is a stochastic matrix. Consider the following general theorem on the convergence of the matrix sequence $\{P^{(k)}\}_{k=0}^{+\infty}$, where $P^{(k)} = P \cdot P \cdot \cdots \cdot P$ (k times).

Theorem 2.1 (e.g., cf. [3], [11]): Let $P=(P(n,m))_{n\geq 0,\, m\geq 0}$ be the transition matrix of a Markov chain. Then, the sequence $\{P^{(k)}\}_{k=0}^{+\infty}$ converges in the Cesaro mean. More precisely, the sequence $\{Q_k\}_{k=1}^{+\infty}$ defined by

$$Q_k=\frac{P+P^{(2)}+\cdots+P^{(k)}}{k}$$

converges to the matrix $Q=\{q(n,m)\}_{n\geq 0,\, m\geq 0}$ with

$$q(n,m)=\frac{\rho(n,m)}{\mu_m},$$

where $\rho(n,m)$ is the probability that starting from the state n the system will ever pass through m, and μ_m is the mean of the real variable which gives the time of return for the first time to the state m, starting from m.

It is obvious that the particular matrix given by (2) is the transition matrix of a Markov chain whose state space is $\mathbb{N}=\{0,1,2,\ldots\}$. We observe that $0,1,\ldots,r-1$ are absorbing states and the other states $r,r+1,\ldots$ are transient, because starting from a state $n\geq r$ the process will be absorbed with probability 1 by one of the states $0,1,\ldots,r-1$ after $n-r+1$ transitions. If m is a transient state, we have $\mu_m=+\infty$ (cf. [3]), hence $q(n,m)=0$ for $m=r,r+1,\ldots$. If n and m are absorbing states, we have $\mu_m=1$ and $\rho(n,m)=\delta_{n,m}$. Therefore, the limit matrix Q of Theorem 2.1 has the following form:

$$Q=\begin{pmatrix} & I_r & & \mid & 0 & \\ \rho(r,0) & \cdots & \rho(r,r-1) & & 0 & \cdots \\ \rho(r+1,0) & \cdots & \rho(r+1,r-1) & & 0 & \cdots \\ \vdots & & \vdots & & \vdots & \\ \rho(n,0) & \cdots & \rho(n,r-1) & & 0 & \cdots \\ \vdots & & \vdots & & \vdots & \end{pmatrix}. \tag{3}$$

The sequence defined by (1) may be written in the following form,

$$X=PX. \tag{4}$$

From expression (4) we derive easily that $X=P^{(n)}X$ for $n\geq 1$, which is equivalent to $X=Q_nX$ for $n\geq 1$, where $Q_n=\frac{P+P^{(2)}+\cdots+p^{(n)}}{n}$. Thus, we have $X=QX$, where Q is given by (3). We then derive the following result.

Theorem 2.2: Let $\{V_n\}_{n=0}^{+\infty}$ be a sequence (1) such that the real numbers $a_0,\ldots,a_{r-1}$ are non-negative with $\sum_{i=1}^{r-1}a_i=1$. Then, for any $n\geq r$, we have

$$V_n=\rho(n,0)V_0+\rho(n,1)V_1+\cdots+\rho(n,r-1)V_{r-1}. \tag{5}$$

Note that the number $\rho(n,j)$ $(0\leq j\leq r-1)$ is the probability of absorption of the process by the state j, starting from the state n. Theorem 2.2 gives the expression of the general term V_n for $n\geq r$ as a function of the initial conditions $V_0,\ldots,V_{r-1}$ and the absorption probabilities $\rho(n,j)$.

2.2 Combinatoric Expression of $\rho(n,m)$

For $n>m\geq r$, the number $\rho(n,m)$ is the probability to reach the state m starting from the state n, because m is a transient state. To reach the state m starting from the state n, the process makes k_j jumps of $j+1$ units with the probability a_j $(0 \le j \le r-1)$. The total number of jumps is $k_0+k_1+\cdots+k_{r-1}$ and the number of units is $k_0+2k_1+\cdots+rk_{r-1}=n-m$. The number of ways to choose the k_j $(0 \le j \le r-1)$ is

$$\frac{(k_0+k_1+\cdots+k_{r-1})!}{k_0!k_1!\cdots k_{r-1}!}$$

and the probability for each choice is $a_0^{k_0}a_1^{k_1}\cdots a_{r-1}^{k_{r-1}}$. Hence, we have the following result.

Theorem 2.3: For any two states n, m $(n > m \ge r)$, the probability $\rho(n,m)$ to reach m starting from n is given by

$$\rho(n,m) = \sum_{k_0+2k_1+\cdots+rk_{r-1}=n-m} \frac{(k_0+k_1+\cdots+k_{r-1})!}{k_0!k_1!\cdots k_{r-1}!} a_0^{k_0}a_1^{k_1}\cdots a_{r-1}^{k_{r-1}}. \tag{6}$$

Note that, for $n > m \ge r$, $\rho(n,m) = H_{n-m+1}^{(r)}(a_0,\dots,a_{r-1})$, where $\{H_{n-m+1}^{(r)}(a_0,\dots,a_{r-1})\}_{n\ge 0}$ is the sequence of multivariate Fibonacci polynomials of order r of Philippou (cf. [1]).

Let n and j be two states such that $0 \le j < r \le n$. Then n is a transient state, j is an absorbing one, and $\rho(n, j)$ is the probability of absorption of the process by j starting from n. First, we suppose that $n \ge 2r$ and $j = 0$. To reach 0 starting from n, the state r is the last transient state visited by the process. And a_{r-1} is the probability of the jump from r to 0, which implies that we have $\rho(n,0) = a_{r-1}\rho(n,r)$. More precisely, to reach j $(0 \le j \le r-1)$ starting from n $(n \ge 2r)$, the process must visit one of the following states $r, r+1, \dots, r+j$, because they are the only states from which the process can reach j in one jump. As $a_{r+k-j-1}$ $(0 \le k \le j)$ is the probability to go from $r+k$ to j and $\rho(n, r+k)$ is the probability to go from n to $r+k$, we obtain

$$\rho(n,j) = a_{r-j-1}\rho(n,r) + a_{r-j}\rho(n,r+1) + \cdots + a_{r-1}\rho(n,r+j). \tag{7}$$

From expression (6), we deduce that $\rho(n,r+l) = \rho(n-l,r)$ for any $n > r+l$. Thus, for any $n \ge 2r$ and j $(0 \le j < r)$, we have

$$\rho(n,j) = a_{r-j-1}\rho(n,r) + a_{r-j}\rho(n-1,r) + \cdots + a_{r-1}\rho(n-j,r).$$

Now suppose that $n < 2r$. Then we have two cases. If $r+j \le n$, expression (7) is still verified. For the second case, $r \le n < r+j$, we have

$$\rho(n,j) = a_{r-j-1}\rho(n,r) + a_{r-j}\rho(n-1,r) + \cdots + a_{n-j-1}\rho(r,r).$$

Hence, the expression of the absorption probabilities is given by Theorem 2.4.

Theorem 2.4: Let n and j be two states such that $0 \le j < r \le n$. Then, if we set $\rho(i,i) = 1$ and $\rho(i,k) = 0$ if $i < k$, the probability of absorption $\rho(n, j)$ is given by

$$\rho(n,j) = a_{r-j-1}\rho(n,r) + a_{r-j}\rho(n-1,r) + \cdots + a_{r-1}\rho(n-j,r), \tag{8}$$

where $\rho(n,0) = a_{r-1}\rho(n,r)$.

2.3 Combinatoric Expression of V_n

By substituting expression (8) in (5), we obtain the following result.

Theorem 2.5: Let $\{V_n\}_{n=0}^{+\infty}$ be a sequence (1). Suppose that the coefficients $a_0,\dots,a_{r-1}$ are nonnegative with $\sum_{i=0}^{r-1} a_i = 1$. Then, for any $n \ge r$, we have

$$V_n = A_0\rho(n,r) + A_1\rho(n-1,r) + \cdots + A_{r-1}\rho(n-r+1,r), \tag{9}$$

where $A_m = a_{r-1}V_m + \cdots + a_m V_{r-1}$; $m = 0, 1, \ldots, r-1$ and the $\rho(k,r)$ are given by (6) with $\rho(r,r) = 1$ and $\rho(k,r) = 0$ if $k < r$.

If we take $V_0 = 1$ and $V_1 = \cdots = V_{r-1} = 0$, we get $V_n = a_{r-1}\rho(n,r)$. Therefore, the sequence $\{(\rho(n,r)\}_{n=0}^{+\infty}$ satisfies the following relation:

$$\rho(n+1,r) = a_0\rho(n,r) + a_1\rho(n-1,r) + \cdots + a_{r-1}\rho(n-r+1,r). \tag{10}$$

Relation (10) may be proved otherwise by considering the jumps of the process from the state $n+1$ to the state r.

2.4 General Case and Levesque Result

Now suppose that the coefficients $a_0, \ldots, a_{r-1}$ are arbitrary real numbers and define the number $\rho(n,r)$ by (6). Then we can prove by induction on n that expression (10) is satisfied. Hence, Theorem 2.5 is still valid in this general case. Such a result was established by Levesque in [5].

3. ASYMPTOTIC BEHAVIOR OF $\rho(n, r)$

Let $\{V_n\}_{n=0}^{+\infty}$ be a sequence (1). Suppose that $a_0, \ldots, a_{r-1}$ are nonnegative real numbers with $\sum_{i=0}^{r-1} a_i = 1$. We have established, using some Markov chains properties, that sequence (1) converges for any $V_0, \ldots, V_{r-1}$ if and only if $\mathrm{CGD}\{i+1; a_i > 0\} = 1$ (cf. [6], Theorem 2.2). When this condition is satisfied, we obtain

$$\lim_{n\to+\infty} V_n = \Pi(0)V_{r-1} + \Pi(1)V_{r-2} + \cdots + \Pi(r-1)V_0, \tag{11}$$

where

$$\Pi(m) = \frac{\sum_{j=m}^{r-1} a_j}{\sum_{i=0}^{r-1}(i+1)a_i} \quad \text{(cf. [6], Theorem 2.4).}$$

By using expressions (9) and (11), we derive the following result.

Theorem 3.1: Suppose that the real numbers $a_0, \ldots, a_{r-1}$ are nonnegative with $\sum_{i=0}^{r-1} a_i = 1$. Then, if $\mathrm{CGD}\{i+1; a_i > 0\} = 1$, we have

$$\lim_{n\to+\infty} \rho(n,r) = \frac{1}{\sum_{j=0}^{r-1}(j+1)a_j},$$

where $\rho(n,r)$ is given by (6).

Now suppose $\sum_{i=0}^{r-1} a_i \neq 1$. It was shown in [7] that under the condition $\mathrm{CGD}\{i+1; a_i > 0\} = 1$, the characteristic equation $x^r = a_0x^{r-1} + \cdots + a_{r-2}x + a_{r-1}$ of sequence (1) has a unique simple nonnegative root q, and the moduli of all other roots is less than q. If we set $b_i = a_i / q^{i+1}$, we have $b_i \geq 0$, $\sum_{i=0}^{r-1} b_i = 1$, and $\mathrm{CGD}\{i+1; a_i > 0\} = \mathrm{CGD}\{i+1; b_i > 0\}$. Thus, the sequence $\{W_n\}_{n=0}^{+\infty}$ defined by $W_n = V_n / q^{n+1}$ is a sequence (1) whose initial conditions are $W_i = V_i / q^{i+1}$ for $i = 0, 1, \ldots, r-1$ and $W_{n+1} = \sum_{i=0}^{r-1} b_i W_{n-i}$ for $n \geq r-1$ (cf. [6]). Therefore, we derive from Theorem 3.1 the following result.

Theorem 3.2: Suppose that $a_0, \ldots, a_{r-1}$ are nonnegative real numbers and $\text{CGD}\{i+1; a_i > 0\} = 1$. Then we have

$$\lim_{n\to+\infty} \frac{1}{q^n}\rho(n,r) = \frac{q^{-r}}{\sum_{j=0}^{r-1}(j+1)\frac{a_j}{q^{j+1}}},$$

where $\rho(n,r)$ is given by (6).

From Theorem 3.2, we can derive a new approximation of the number q. More precisely, we have the following corollary.

Corollary 3.3: Suppose that $a_0, \ldots, a_{r-1}$ are nonnegative real numbers and $\text{CGD}\{i+1; a_i > 0\} = 1$. Then the unique simple nonnegative root q of the characteristic equation of (1) is given by

$$q = \lim_{n\to+\infty} \sqrt[n]{\rho(n,r)},$$

where $\rho(n,r)$ is given by (6).

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