A GENERALIZATION OF THE EULER AND JORDAN TOTIENT FUNCTIONS

Temba Shonhiwa
Dept. of Mathematics, The University of Zimbabwe, PO Box MP 167, Mt. Pleasant, Harare, Zimbabwe
(Submitted April 1997-Final Revision July 1997)

1. THE FUNCTION $S_m^k(n)$ AND RELATED RESULTS

This article was motivated by a question posed to me by Professor H. W. Gould [2], specifically: What can be said about the number theoretic function

$$G_m(n) = \sum_{\substack{1\le a_1,\ldots,a_m\le n \\ (a_1,\ldots,a_m)=1}} 1, \quad \text{where } m\ge 2, n\ge 1? \tag{1}$$

The Jordan totient function $J_m(n)$ generalizes Euler's totient function $\phi(n)$. In this paper we investigate the function $S_m^k(n)$, which generalizes both Jordan and Euler's totient functions. Thus, with $m\ge 1$, $n\ge 1$, and $k\ge 1$, let

$$S_m^k(n) = \sum_{\substack{1\le a_1,\ldots,a_m\le n \\ (a_1,\ldots,a_m,k)=1}} 1 . \tag{2}$$

The case $k=n$ retrieves $J_m(n)$, while $S_1^n(n)$ is Euler's totient function. Also, it is clear that $S_m^1(n)=n^m=I_m(n)$. In fact, $\sigma_m(n)=\Sigma_{d|n} S_m^1(d)$, from which we obtain by Möbius inversion that

$$S_m^1(n)=\sum_{d|n}\mu(d)\sigma_m\left(\frac{n}{d}\right).$$

Also, since $\Sigma_{d|n} J_m(d) S_m^1(n)$, it follows that

$$J_m(n)=\sum_{d|n}\mu(d)S_m^1\left(\frac{n}{d}\right)=n^m\sum_{d|n}\frac{\mu(d)}{d^m} \quad\text{and}\quad \sigma_m(n)=\sum_{d|n}\sum_{t|d}J_m(t).$$

We shall make use of the following known result.

Theorem 1: Let $f(n)$ and $F(n)$ be number theoretic functions such that $F(n)=\Sigma_{d|n} f(d)$. Then, for any integer N,

$$\sum_{n=1}^{N}F(n)=\sum_{n=1}^{N}\sum_{d|n}f(d)=\sum_{j=1}^{N}f(j)\left[\frac{N}{j}\right].$$

We may use this theorem to obtain the result that

$$\sum_{j=1}^{n}S_m^1(j)=\sum_{j=1}^{n}j^m=\sum_{j=1}^{n}\sum_{d|n}J_m(d)=\sum_{j=1}^{n}\left[\frac{n}{j}\right]J_m(j). \tag{3}$$

We now prove our next result.

Theorem 2: Let $k=\prod_{i=1}^{s}p_i^{e_i}$ be the prime decomposition of k, where $e_i\ge 1$, then

$$S_m^k(n)=\sum_{d|k}\mu(d)\left[\frac{n}{d}\right]^m.$$

Proof: It follows by the inclusion-exclusion theorem that

$$S_m^k(n) = n^m - \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{\substack{i_m=1\\ p_i|(i_1,\dots,i_m,k)\\ 1\le i\le s}}^{n} 1 + \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{\substack{i_1=1\\ p_ip_j|(i_1,\dots,i_m,k)\\ 1\le i<j\le s}}^{n} 1 + \cdots + (-1)^s\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{\substack{i_m=1\\ p_1p_2\cdots p_s|(i_1,\dots,i_m,k)}}^{n} 1$$

$$= n^m - \sum_{1\le i_m\le\left[\frac{n}{p_i}\right]}\sum_{1\le i_m\le\left[\frac{n}{p_i}\right]}\cdots\sum_{1\le i_m\le\left[\frac{n}{p_i}\right]} 1 + \sum_{1\le i_1\le\left[\frac{n}{p_ip_j}\right]}\sum_{1\le i_2\le\left[\frac{n}{p_ip_j}\right]}\cdots\sum_{1\le i_m\le\left[\frac{n}{p_ip_j}\right]} 1 + \cdots + (-1)^s\sum_{1\le i_1\le\left[\frac{n}{p_1\cdots p_s}\right]}\cdots\sum_{1\le i_m\le\left[\frac{n}{p_1\cdots p_s}\right]} 1$$

$$= n^m - \left[\frac{n}{p_i}\right]^m + \left[\frac{n}{p_ip_j}\right]^m + \cdots + (-1)^s\left[\frac{n}{p_1p_2\cdots p_s}\right]^m = \sum_{d|k}\mu(d)\left[\frac{n}{d}\right]^m,$$

where the subindices are as defined in the first line.

For the special case of $k = n$ and $m = 1$, it follows that

$$\phi(n) = n - \frac{n}{p_i} + \frac{n}{p_ip_j} + \cdots + (-1)^s\frac{n}{p_1\cdots p_s} = n\prod_{p|n}\left(1-\frac{1}{p}\right) = n\sum_{d|n}\frac{\mu(d)}{d},$$

as expected. Also

$$J_m(n) = n^m - \left(\frac{n}{p_i}\right)^m + \left(\frac{n}{p_ip_j}\right)^m + \cdots + (-1)^s\left(\frac{n}{p_1\cdots p_s}\right)^m = n^m\prod_{p|n}\left(1-\frac{1}{p^m}\right) = n^m\sum_{d|n}\frac{\mu(d)}{d^m}$$

again as expected.

Similarly, it may be shown that

$$S_m^n(n^\alpha) = n^{m\alpha}\prod_{p|n}\left(1-\frac{1}{p^m}\right) = n^{m\alpha}\sum_{d|n}\frac{\mu(d)}{d^m}. \tag{4}$$

Further, by setting , we obtain the result,

$$S_1^k(n) = \sum_{\substack{1\le i\le n\\ (i,k)=1}} 1 = \sum_{d|k}\mu(d)\left[\frac{n}{d}\right].$$

On the other hand, by defining

$$S_\tau^k(n) = \sum_{\substack{d|n\\ (d,k)=1}} 1,$$

we obtain the following result.

Theorem 3: $S_\tau^k(n) = \sum_{\substack{d|n\\ (d,k)=1}} 1 = \sum_{d|(k,n)}\mu(d)\tau\left(\frac{n}{d}\right).$

We may generalize the function $S_m^k(n)$ by setting

$$S_m^k(n,a) = \sum_{\substack{1\le a_1,\dots,a_m\le n\\ (a_1,\dots,a_m,k)=a}} 1 = \sum_{\substack{1\le b_1,\dots,b_m\le\left[\frac{n}{a}\right]\\ \left(b_1,\dots,b_m,\frac{k}{a}\right)=1}} 1 = \begin{cases} S_m^{a/k}\left(\left[\frac{n}{a}\right]\right), & \text{if } a/k,\\ 0, & \text{otherwise.}\end{cases}$$

We now let $S_1^k(x)$ denote the generating function for $S_1^k(n)$, then

$$S_1^k(x) = \sum_{n=1}^{\infty} S_1^k(n)\,x^n = \sum_{n=1}^{\infty} nx^n - \sum_{n=1}^{\infty}\left[\frac{n}{p_i}\right]x^n + \sum_{n=1}^{\infty}\left[\frac{n}{p_ip_j}\right]x^n + \cdots + (-1)^s\sum_{n=1}^{\infty}\left[\frac{n}{p_1p_2\cdots p_s}\right]x^n$$

$$= \sum_{n=1}^{\infty} nx^n - x^{p_i}\sum_{n=p_i}^{\infty}\left[\frac{n}{p_i}\right]x^{n-p_i} + \cdots + (-1)^s x^{p_1\cdots p_s}\sum_{n=p_1\cdots p_s}^{\infty}\left[\frac{n}{p_1p_2\cdots p_s}\right]x^{n-p_1\cdots p_s}$$

$$= \frac{x}{(1-x)^2} - \frac{x^{p_i}}{(1-x)(1-x^{p_i})} + \frac{x^{p_ip_j}}{(1-x)(1-x^{p_ip_j})} + \cdots + (-1)^s\frac{x^{p_i\cdots p_s}}{(1-x)(1-x^{p_1\cdots p_s})},$$

where we have used the result,

$$\sum_{n=k}^{\infty}\left[\frac{n}{k}\right]x^{n-k} = \frac{1}{(1-x)(1-x^k)},\quad |x|<1.$$

We now use Theorem 2 to partially answer Gould's question, as follows.

Theorem 4: Let $G_m(n) = \sum_{\substack{1\le a_1,\ldots,a_m\le n\\(a_1,\ldots,a_m)=1}} 1$, where $m\ge 2$, $n\ge 1$. Then

$$G_m(n) = \sum_{k=1}^{n}\sum_{\substack{1\le a_1,\ldots,a_{m-1}\le n\\(a_1,\ldots,a_{m-1},k)=1}} 1 = \sum_{k=1}^{n} S_{m-1}^k(n) = \sum_{k=1}^{n}\sum_{d\mid k}\mu(d)\left[\frac{n}{d}\right]^{m-1},\text{ by Theorem 2.}$$

We now restrict the function $S_m^k(n)$ somewhat and define a new function thus:

$$L_m^k(n) = \sum_{\substack{1\le a_1\le a_2\le\cdots\le a_m\le n\\(a_1,\ldots,a_m,k)=1}} 1,\quad\text{where } m\ge 1,\ n\ge 1,\ k\ge 1. \tag{5}$$

The case $k=1$ gives the following result.

Theorem 5: $L_m^1(n) = \binom{n+m-1}{m}$.

Proof: We prove the result by induction on m. First of all, the case $m=2$ gives

$$L_2^1(n) = \sum_{\substack{1\le a\le b\le n\\(a,b,1)=1}} 1 = \sum_{i=1}^{n}\sum_{j=1}^{i} 1 = \frac{n^2}{2} + \frac{n}{2} = \binom{n+2-1}{2}.$$

We now assume the result true for 1, 2, 3, ..., m and consider

$$L_{m+1}^1(n) = \sum_{i_1=1}^{n}\sum_{i_2=1}^{i_1}\cdots\sum_{i_{m+1}=1}^{i_m} 1 = \sum_{i_1=1}^{n} L_m^1(i_1) = \sum_{j=1}^{n}\binom{j+m-1}{m}.$$

Now let $j' = j+m-1$. After reverting back to the original variable, we obtain

$$L_{m+1}^1(n) = \sum_{j=m}^{j+n-1}\binom{j}{m} = \binom{m+n}{m+1},$$

see Gould [3, (1.52), p. 7], and hence, the induction goes through.

Alternatively, we may show that

$$L_m^1(n) = \frac{\sum_{i=0}^{m-1} S_{m-i}^1(n) S_1(m-1, i)}{\sum_{i=0}^{m-1} S_1(m-1, i)},$$

where $S_1(m, i)$ represents Stirling numbers of the first kind in Gould's notation [4]. We note that $s(n, m) = (-1)^{n-m} S_1(n-1, n-m)$, where $s(n, m)$ represents Stirling numbers of the first kind in Riordan's notation [6]. The equivalence follows from the fact that

$$\sum_{i=0}^{m-1} S_{m-i}^1(n) S_1(m-1, i) = \sum_{i=1}^{m} S_1(m-1, m-i) n^i$$
$$= \sum_{i=1}^{m} (-1)^{i-m} s(m, i) n^i = (-1)^m m! \binom{-n}{m} = m! \binom{n+m-1}{m}.$$

And, of course,

$$\sum_{i=0}^{m-1} S_1(m-1, i) = (-1)^i m! \binom{-1}{m} = m!.$$

From Theorem 5 and the standard result,

$$F_{n+1} = \sum_{i=0}^{[\frac{n}{2}]} \binom{n-j}{j},$$

where the F_n are Fibonacci numbers, we may deduce that

$$F_n = \sum_{j=0}^{[\frac{n-1}{2}]} \binom{n-1-j}{j} = \sum_{j=1}^{[\frac{n-1}{2}]+1} \binom{n-j}{j-1} = \sum_{j=1}^{[\frac{n+1}{2}]} \binom{n-j}{n-2j+1} = \sum_{j=1}^{[\frac{n+1}{2}]} L_{(n-2j+1)}^1(j),$$

where $L_0^1(n) = 1 \forall n$.

We now let $k = \prod_{i=1}^{s} p_i^{e_i}$, where $e_i \geq 1$, and prove our next result.

Theorem 6: $L_m^k(n) = \sum_{d|k} \mu(d) L_m^1\left(\left[\frac{n}{d}\right]\right).$

Proof:

$$L_m^k(n) = L_m^1(n) - \sum_{i_1=1}^{n} \sum_{i_2=1}^{i_1} \cdots \sum_{\substack{i_m=1 \\ p_i|(i_1,\ldots,i_m,k)}}^{i_{m-1}} 1 + \sum_{i_1=1}^{n} \sum_{i_2=1}^{i_1} + \cdots + \sum_{\substack{i_m=1 \\ p_i p_j|(i_1,\ldots,i_m,k)}}^{i_{m-1}} 1$$
$$+ \cdots + (-1)^s \sum_{i_1=1}^{n} \sum_{i_2=1}^{i_1} \cdots \sum_{\substack{i_m=1 \\ p_1 p_2 \cdots p_s|(i_1,\ldots,i_m,k)}}^{i_{m-1}} 1$$
$$= L_m^1(n) - L_m^1\left(\left[\frac{n}{p_i}\right]\right) + L_m^1\left(\left[\frac{n}{p_i p_j}\right]\right) + \cdots + (-1)^s L_m^1\left(\left[\frac{n}{p_1 p_2 \cdots p_s}\right]\right) = \sum_{d|k} \mu(d) L_m^1\left(\left[\frac{n}{d}\right]\right).$$

The special case $k = n$ gives the result

$$L_m^n(n) = \sum_{d|n} \mu(d) L_m^1\left(\frac{n}{d}\right) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \binom{d+m-1}{m},$$

which implies that $L_m^1(n) = \sum_{d|n} L_m^d(d)$. Equivalently,

$$\sum_{n=1}^{\infty} L_m^n(n)\frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} L_m^1(n)x^n = \frac{x}{(1-x)^{m+1}} \tag{6}$$

or

$$\zeta(s)\sum_{n=1}^{\infty}\frac{L_m^n(n)}{n^s} = \sum_{n=1}^{\infty}\frac{L_m^1(n)}{n^s}.$$

It follows from Theorem 1 that

$$\sum_{j=1}^{n} L_m^1(j) = \sum_{j=1}^{n}\sum_{d|j} L_m^d(d) = \sum_{j=1}^{n}\left[\frac{n}{j}\right]L_m^j(j),$$

that is,

$$\sum_{j=1}^{n}\binom{m+j-1}{m} = \sum_{j=1}^{n}\left[\frac{n}{j}\right]L_m^j(j)$$

which, on letting $m+j-1=j'$ and reverting back to the original variable, gives

$$\sum_{j=m}^{m+n-1}\binom{j}{m} = \binom{m+n}{m+1} = \sum_{j=1}^{n}\left[\frac{n}{j}\right]L_m^j(j). \tag{7}$$

The case $m=1$ gives the result $L_1^n(n) = \phi(n)$. Following are the tables of the values of the $L_m^n(n)$ and $L_m^1(n)$ arrays.

TABLE 1. Values of the $L_m^n(n)$ Array

m \ n	1	2	3	4	5	6	7	8
1	1	1	2	2	4	2	6	4
2	1	2	5	7	14	13	27	26
3	1	3	9	16	34	43	83	100
4	1	4	14	30	69	107	209	295
5	1	5	20	50	125	226	461	736
6	1	6	27	77	209	428	923	1632
7	1	7	35	112	329	749	1715	3312
8	1	8	44	156	494	1234	3002	6270

TABLE 2. Values of the $L_m^1(n)$ Array

m \ n	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	1	3	6	10	15	21	28	36
3	1	4	10	20	35	56	84	120
4	1	5	15	35	70	126	210	330
5	1	6	21	56	126	252	462	792
6	1	7	28	84	252	462	924	1716
7	1	8	36	120	462	924	1716	3432
8	1	9	45	165	792	1716	3003	6435

We obtain a recurrence relation for $L_m^1(n)$ as follows:

$$L_m^1(n+1) = \binom{m+n}{m} = \binom{m+n-1}{m} + \binom{n+m-1}{m-1} = L_m^1(n) + L_{m-1}^1(n+1).$$

With the exception of boundary conditions, we note that this relation is the same as equation (1.1) of Carlitz and Riordan [1]. Its generating function $L_m^1(x)$ is

$$\begin{aligned} L_m^1(x) &= \sum_{n=1}^{\infty} L_m^1(n)x^n = \sum_{n=1}^{\infty} L_m^1(n+1)x^n - \sum_{n=1}^{\infty} L_{m-1}^1(n+1)x^n \\ &= \sum_{n=2}^{\infty} L_m^1(n)x^{n-1} - \sum_{n=2}^{\infty} L_{m-1}^1(n)x^{n-1}, \end{aligned}$$

which implies that

$$xL_m^1(x) = \sum_{n=1}^{\infty} L_m^1(n)x^n - xL_m^1(1) - \sum_{n=1}^{\infty} L_{m-1}^1(n)x^n + xL_{m-1}^1(1),$$

that is,

$$L_m^1(x) = \frac{L_{m-1}^1(x)}{1-x} = \frac{L_1^1(x)}{(1-x)^{m-1}}.$$

But

$$L_1^1(x) = \sum_{n=1}^{\infty} L_1^1(n)x^n = \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$$

and, therefore,

$$L_m^1(x) = \frac{x}{(1-x)^{m+1}}, \; |x| < 1.$$

We may also let

$$S_n = \sum_{m=1}^{n} L_m^1(n) = \sum_{m=1}^{n} \binom{n+m-1}{m} = \sum_{m=0}^{n} \binom{n+m-1}{m} - \binom{n-1}{0} = \binom{2n}{n} - 1.$$

Similarly, we may define and show that

$$T_m = \sum_{n=1}^{m} \binom{n+m-1}{m} = \sum_{j=m}^{2m-1} \binom{j}{m} = \binom{2m}{m+1}.$$

We now seek the generating function of S_n. And so, with $S_0 = 0$, let

$$S(x) = \sum_{n=0}^{\infty} S_n x^n = \sum_{n=0}^{\infty} \left\{ \binom{2n}{n} - 1 \right\}.$$

We now use the result

$$\binom{-\frac{1}{2}}{n} = (-1)^n \binom{2n}{n} 2^{-2n}, \; n \geq 0,$$

to obtain

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \sum_{n=0}^{\infty} (-1)^n \binom{\frac{-1}{2}}{n} 2^{2n} x^n = (1-4x)^{\frac{-1}{2}};$$

hence,

$$S(X) = \frac{1}{\sqrt{1-4x}} - \frac{1}{1-x}$$

see Gould [4, p. 16].

Finally, we may consider the function

$$T_m^k(n) = \sum_{\substack{1 \le a_1 < a_2 < \cdots < a_m \le n \\ (a_1, \ldots, a_m, k) = 1}} 1 \quad , \quad n \ge m. \tag{8}$$

The case $k = 1$ gives

$$\begin{aligned} T_m^1(n) &= \sum_{i_1=1}^{n-(m-1)} \sum_{i_2=i_1+1}^{n-(m-2)} \cdots \sum_{i_m=i_{m-1}+1}^{n} 1 = \frac{\sum_{i=1}^m s(m,i)n^i}{\sum_{i=1}^m |s(m,i)|} \\ &= \frac{\sum_{i=0}^m s(m,i)n^i}{\sum_{i=0}^{m-1} S_1(m-1,i)} = \frac{m!\binom{n}{m}}{m!} = \binom{n}{m}. \end{aligned} \tag{9}$$

This is a known result.

2. INVERSE AND ORTHOGONAL RELATIONS

Using Theorem 2, we may now prove our next result.

Theorem 7: $T_m^k(n) = \sum_{d|k} \mu(d) T_m^1\left(\left[\frac{n}{d}\right]\right).$

The case $k = n$ gives

$$T_m^n(n) = \sum_{d|n} \mu\binom{n}{d}\binom{d}{m},$$

from which it follows that $T_m^n(n) = \phi(n)$ and $T_m^m(m) = 1$. Möbius inversion then gives

$$\binom{n}{m} = \sum_{d|n} T_m^d(d);$$

hence,

$$\sum_{n=1}^{\infty} \frac{T_m^n(n)x^n}{1-x^n} = \sum_{n=1}^{\infty} T_m^1(n)x^n = x^m \sum_{n=m}^{\infty} \binom{n}{m} x^{n-m} = \frac{x^m}{(1-x)^{m+1}} \tag{10}$$

or

$$\zeta(s) \sum_{n=1}^{\infty} \frac{T_m^n(n)}{n^s} = \sum_{n=1}^{\infty} \frac{T_m^1(n)}{n^s}.$$

It follows from Theorem 1 that

$$\sum_{j=m}^{n} \binom{j}{m} = \binom{n+1}{m+1} = \sum_{j=m}^{n} \sum_{d|j} T_m^d(d) = \sum_{j=m}^{n} \left[\frac{n}{j}\right] T_m^j(j). \tag{11}$$

Following a technique of Gould [5, p. 252], we may set

$$\left[\frac{j}{m}\right] = \sum_{i=m}^{j} \binom{j+1}{i+1} K_m(i);$$

hence,

$$\sum_{j=m}^{n}(-1)^{n-j-1}\binom{n}{j+1}\left[\frac{j}{m}\right]=\sum_{j=m}^{n}\sum_{i=m}^{j}(-1)^{n-j-1}\binom{n}{j+1}\binom{j+1}{i+1}K_m(i)$$

$$=\sum_{i=m}^{n}\sum_{j=i}^{n}(-1)^{n-j-1}\binom{n}{j+1}\binom{j+1}{i+1}K_m(i)=\sum_{i=m}^{n}K_m(i)\sum_{j=i+1}^{n+1}(-1)^{n-j}\binom{n}{j}\binom{j}{i+1}=K_m(n-1).$$

From this, we may obtain the inverse to $T_m^n(n)$ as

$$K_m(n)=\sum_{j=m}^{n}(-1)^{n-j}\binom{n+1}{j+1}\left[\frac{j}{m}\right]. \tag{12}$$

Also, since

$$\sum_{n=1}^{\infty}\frac{T_m^n(n)x^n}{1-x^n}=\frac{x^m}{(1-x)^{m+1}},$$

we may consider

$$\sum_{j=1}^{\infty}K_m(j)\frac{x^j}{(1-x)^{j+1}}=\sum_{j=m}^{\infty}\sum_{i=m}^{j}(-1)^{j-i}\binom{j+1}{i+1}\left[\frac{i}{m}\right]\frac{x^j}{(1-x)^{j+1}}$$

$$=\frac{1}{1-x}\sum_{i=m}^{\infty}(-1)^i\left[\frac{i}{m}\right]\sum_{j=i}^{\infty}\binom{j+1}{i+1}\left(\frac{x}{x-1}\right)^j.$$

But

$$\sum_{n=1}^{\infty}\binom{n+1}{m+1}x^n=\sum_{n=1}^{\infty}\sum_{j=m}^{n}\binom{j}{m}x^n=\sum_{j=1}^{\infty}\binom{j}{m}\sum_{n=j}^{\infty}x^n=\frac{x^m}{1-x}\sum_{j=m}^{\infty}\binom{j}{m}x^{j-m}=\frac{x^m}{(1-x)^{m+2}}.$$

Therefore,

$$\sum_{j=i}^{\infty}\binom{j+i}{i+1}\left(\frac{x}{x-1}\right)^j=\left(\frac{x}{x-1}\right)^i(1-x)^{i+2}$$

and so

$$\sum_{j=1}^{\infty}K_m(j)\frac{x^j}{(1-x)^{j+1}}=\frac{(x-1)^2x^m}{1-x}\sum_{i=m}^{\infty}\left[\frac{i}{m}\right]x^{i-m}=\frac{x^m}{1-x^m}. \tag{13}$$

From equations (10) and (13), we obtain the following result.

Theorem 8: The functions $K_m(n)$ and $T_m^n(n)$ satisfy the orthogonality relations

$$\sum_{j=m}^{n}T_m^j(j)K_j(n)=\delta_m^n \quad\text{and}\quad \sum_{j=m}^{n}K_m(j)T_j^n(n)=\delta_m^n.$$

Therefore, we have the following general inversion result.

Theorem 9: For any ordered function sequence pair, $\langle f(n,m), g(n,m)\rangle$,

$$f(n,m)=\sum_{j=m}^{n}g(n,j)T_m^j(j) \text{ if and only if } g(n,m)=\sum_{j=m}^{n}f(n,j)K_m(j).$$

The ordered function pair

$$\left\langle \binom{n+1}{m+1}, \left[\frac{n}{m}\right] \right\rangle$$

is a particular case of this theorem.

We also note the following concerning $T_m^n(n)$:

$$\sum_{j=1}^{n} T_j^n(n) = \sum_{j=1}^{n} \sum_{d|n} \mu(d) \binom{\frac{n}{d}}{j} = n \sum_{j=1}^{n} j \sum_{d|n} \frac{\mu(d)}{d} \binom{\frac{n}{d}-1}{j-1},$$

that is, $\sum_{j=1}^{n} T_j^n(n)$ is divisible by n. Furthermore,

$$\sum_{j=1}^{n} T_j^n(n) x^n = \sum_{j=1}^{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) \binom{d}{j} x^n = \sum_{d|n} \mu\left(\frac{n}{d}\right) (x+1)^d,$$

from which we obtain

$$\sum_{j=1}^{n} T_j^n(n) = \sum_{j=1}^{n} \sum_{\substack{1 \le a_1 < a_2 < \cdots < a_j \le n \\ (a_1, \ldots, a_j, n) = 1}} 1 = \sum_{d|n} \mu\left(\frac{n}{d}\right) 2^d.$$

Similarly,

$$\sum_{j=1}^{n} K_j(n) x^j = \sum_{j=1}^{n} \sum_{i=j}^{n} (-1)^{n-i} \binom{n+1}{i+1} \left[\frac{i}{m}\right] x^j = \sum_{i=1}^{n} (-1)^{n-i} \binom{n+1}{i+1} \sum_{j=1}^{i} \left[\frac{i}{m}\right] x^j,$$

From which we obtain

$$\sum_{j=1}^{n} K_j(n) = \sum_{i=1}^{n} (-1)^{n-i} \binom{n+1}{i+1} \sum_{j=1}^{i} \left[\frac{i}{m}\right] = \sum_{j=1}^{n} \tau(j) \sum_{i=j}^{n} (-1)^{n-i} \binom{n+1}{i+1} = \sum_{j=1}^{n} \tau(j)(-1)^{n+j} \binom{n}{j}$$

Inversely, it may be shown that

$$\tau(n) = \sum_{j=1}^{n} \binom{n}{j} \sum_{i=1}^{j} K_i(j),$$

a result similar to one obtained by Gould [5, p. 255]. Following are tables of the arrays of the two functions $T_m^n(n)$ and $K_m(n)$.

TABLE 3. The $T_m^n(n)$ Array

$m \backslash n$	1	2	3	4	5	6	7	8
1	1	1	2	2	4	2	6	4
2	0	1	3	5	10	11	21	22
3	0	0	1	4	10	19	35	52
4	0	0	0	1	5	15	35	69
5	0	0	0	0	1	6	21	56
6	0	0	0	0	0	1	7	28
7	0	0	0	0	0	0	1	8
8	0	0	0	0	0	0	0	1

TABLE 4. The $K_m(n)$ Array

$m \backslash n$	1	2	3	4	5	6	7	8
1	1	−1	1	−1	1	−1	1	−1
2	0	1	−3	7	−15	−4	7	127
3	0	0	1	−4	10	−19	28	−28
4	0	0	0	1	−5	15	−34	71
5	0	0	0	0	1	−6	21	−56
6	0	0	0	0	0	1	−7	28
7	0	0	0	0	0	0	1	−8
8	0	0	0	0	0	0	0	1

ACKNOWLEDGMENT

The author wishes to thank the anonymous referee for valuable comments.

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AMS Classification Numbers: 11A25, 11B75

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