ON THE 2-ADIC VALUATIONS OF THE TRUNCATED POLYLOGARITHM SERIES

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The aim of this paper is to prove the following theorem which was conjectured in [1] and [2] (and originated in a work of Yu [3]).

Theorem 1: Set

$$S_1(N)=\sum_{j=1}^{N}\frac{2^j}{j}.$$

Then, if $\nu(x)$ denotes the highest exponent of 2 that divides x (i.e., the 2-adic valuation), we have

$$\nu(S_1(2^m))=2^m+2m-2 \quad \text{for } m\geq 4.$$

For the sake of completeness, note that a direct computation shows that

$$\nu(S_1(2^m))=2^m+2m+d_1(m),$$

with $d_1(0)=0$, $d_1(1)=-2$, $d_1(2)=-3$, and $d_1(3)=-1$, the theorem claiming that $d_1(m)=-2$ for $m\geq 4$.

Before proving this theorem, we will need a few lemmas. In this paper, we will work entirely in the field $\mathbb{Q}_2$ of 2-adic numbers, on which the valuation ν can be extended.

Lemma 2: We have

$$\sum_{j=1}^{\infty}\frac{2^j}{j}=0 \quad \text{in } \mathbb{Q}_2.$$

Proof: Since the function

$$\mathrm{Li}_1(x)=-\log(1-x)=\sum_{j=1}^{\infty}\frac{x^j}{j}$$

converges in $\mathbb{Q}_2$ for $\nu(x)\geq 1$, and satisfies

$$\mathrm{Li}_1(x)+\mathrm{Li}_1(y)=-\log((1-x)(1-y))=\mathrm{Li}_1(x+y-xy)$$

for all x and y such that $\nu(x)\geq 1$ and $\nu(y)\geq 1$, we deduce that our sum is equal to $\mathrm{Li}_1(2)$ and that

$$2\mathrm{Li}_1(2)=\mathrm{Li}_1(0)=0,$$

so $\mathrm{Li}_1(2)=0$ as claimed. □

Lemma 3: We have

$$\sum_{j=1}^{\infty}\frac{2^j}{j^2}=0 \text{ in } \mathbb{Q}_2.$$

Proof: This time we set

$$\mathrm{Li}_2(x)=\sum_{j=1}^{\infty}\frac{x^j}{j^2}.$$

This is the 2-adic dilogarithm, and converges in $\mathbb{Q}_2$ for $v(x)\geq 1$. Most of the usual complex functional equations for the dilogarithm are still valid in the p-adic case. The one we will need here is the following:

$$\mathrm{Li}_2(x)+\mathrm{Li}_2\left(\frac{-x}{1-x}\right)=-\frac{1}{2}\log^2(1-x),$$

valid for $v(x)>1$. This can be proved by differentiation, or simply by noting that it is a formal identity valid over the field $\mathbb{C}$, hence also over any field of characteristic zero.

Setting $x=2$, we obtain

$$2\mathrm{Li}_2(2)=-\log(-1)^2/2=-\mathrm{Li}_1(2)^2/2=0$$

by Lemma 2, thus proving Lemma 3. □

Remark: Lemmas 2 and 3 cannot be generalized immediately to polylogarithms. For example, an easy computation shows that $\mathrm{Li}_3(2)\neq 0$, and in fact that $v(\mathrm{Li}_3(2))=-2$ (this is the explanation of $d_1(m)=-2$, as we will see below). I do not know if the value (in $\mathbb{Q}_2$) of $\mathrm{Li}_3(2)$ can be computed explicitly. See also Theorem 8 below.

We can now prove the following.

Lemma 4: For all $N\geq 0$, we have

$$S_1(N)=\sum_{j=1}^{N}\frac{2^j}{j}=-N^2 2^N\sum_{j=1}^{\infty}\frac{2^j}{j^2(j+N)}.$$

Proof: From Lemma 2, we deduce that

$$S_1(N)=-\sum_{j=N+1}^{\infty}\frac{2^j}{j}=-\sum_{j=1}^{\infty}\frac{2^{N+j}}{N+j}=-2^N\sum_{j=1}^{\infty}\frac{2^j}{j+N}.$$

Applying Lemma 2 again, we deduce that

$$S_1(N)=S_1(N)+2^N\sum_{j=1}^{\infty}\frac{2^j}{j}=N2^N\sum_{j=1}^{\infty}\frac{2^j}{j(j+N)}.$$

Finally, applying Lemma 3, we obtain

$$S_1(N)=S_1(N)-N2^N\sum_{j=1}^{\infty}\frac{2^j}{j^2}=-N^2 2^N\sum_{j=1}^{\infty}\frac{2^j}{j^2(j+N)}$$

as claimed. □

We can now prove Theorem 1. It follows from Lemma 4 that

$$v(S_1(2^m))=2^m+2m+v(T_1(2^m))\quad\text{with}\quad T_1(2^m)=\sum_{j=1}^{\infty}\frac{2^j}{j^2(j+2^m)}.$$

Thus, Theorem 1 is equivalent to showing that $v(T_1(2^m)) = -2$ for $m \geq 4$. This will immediately follow from Lemma 5.

Lemma 5: Set

$$w_1(j,m) = v\left(\frac{2^j}{j^2(j+2^m)}\right).$$

Then, for $m \geq 4$, we have $w_1(j,m) \geq -1$ for all j except for $j = 4$ for which $w_1(4,m) = -2$.

Since there is a unique term in the sum defining $T_1(2^m)$ having minimal valuation, it follows that the valuation of $T_1(2^m)$ is equal to that minimum; therefore, Theorem 1 clearly follows from Lemma 5.

Proof: Set $j = 2^a i$ with i odd. If $a < m$, we have $w_1(j,m) = 2^a i - 3a \geq 2^a - 3a$, with equality only if $i = 1$. Clearly, the function $2^a - 3a$ attains a unique minimum on the integers for $a = 2$, where its value is equal to -2; hence, if $a < m$, $w_1(j,m) \geq -1$ except for $a = 2$ and $i = 1$, i.e., for $j = 4$ for which $w_1(j,m) = -2$. Note that this value can be attained only if $2 < m$, i.e., if $m \geq 3$.

If $a < m$, we have $w_1(j,m) = 2^a i - 2a - m \geq 2^a i - 3a + 1 \geq -1$ for all j by what we have just proved.

Finally, if $a = m$, we have $w_1(j,m) = 2^m i - 3m - v(i+1)$. We note that, for all i, we have $v(i+1) \leq i$. Thus,

$$w_1(j,m) \geq (2^m - 1)i - 3m \geq 2^m - 3m - 1 \geq -1 \quad \text{for } m \geq 4.$$

Note that this is the only place where it is necessary to assume that $m \geq 4$ (for $m = 3$ the minimum would be -2, so we could not conclude that the valuation of the sum is equal to -2, and in fact it is not). This proves Lemma 5, hence Theorem 1. □

Remark: Lemma 4 and suitable generalizations of Lemma 5 allow us more generally to compute $v(S_1(h2^m))$ for $m \geq 4$ and a fixed odd h. I leave the details to the reader.

In view of Lemma 3, it is natural to ask if there is a generalization of Theorem 1 to the dilogarithm. This is indeed the case.

Theorem 6: Set

$$S_2(N) = \sum_{j=1}^{N} \frac{2^j}{j^2}.$$

Then we have

$$v(S_2(2^m)) = 2^m + m - 1 \quad \text{for } m \geq 4.$$

For the sake of completeness, note that a direct computation shows that

$$v(S_2(2^m)) = 2^m + m + d_2(m),$$

with $d_2(0) = 0$, $d_2(1) = -3$, $d_2(2) = -4$, and $d_2(3) = -3$, the theorem claiming that $d_2(m) = -1$ for $m \geq 4$.

Proof: By Lemma 3, we have

$$S_2(N) = -\sum_{j=N+1}^{\infty} \frac{2^j}{j^2} = -2^N \sum_{j=1}^{\infty} \frac{2^j}{(j+N)^2}.$$

Applying Lemma 3 once again, we have

$$S_2(N) = S_2(N) + 2^N \sum_{j=1}^{\infty} \frac{2^j}{j^2} = N2^N \sum_{j=1}^{\infty} \frac{2^j(2j+N)}{j^2(j+N)^2}.$$

The proof is now nearly identical to that of Theorem 1. We have

$$v(S_2(2^m)) = 2^m + m + v(T_2(2^m)),$$

with

$$T_2(2^m) = \sum_{j=1}^{\infty} \frac{2^j(2j+2^m)}{j^2(j+2^m)^2}.$$

Further, we have

Lemma 7: Set

$$w_2(j,m) = v\left(\frac{2^j(2j+2^m)}{j^2(j+2^m)^2}\right).$$

Then, for $m \geq 4$, we have $w_2(j,m) \geq 0$ for all j except $j=4$ for which $w_2(4,m) = -1$.

Since there is a unique term in the sum defining $T_2(2^m)$ having minimal valuation, it follows as before that the valuation of $T_2(2^m)$ is equal to that minimum; hence, Theorem 6 clearly follows from Lemma 7.

Proof: Set $j = 2^a i$ with i odd. If $a < m-1$, we have $w_2(j,m) = 2^a i - 3a + 1 \geq 2^a - 3a + 1$, with equality only if $i = 1$. The function $2^a - 3a + 1$ attains a unique minimum on the integers for $a = 2$, where its value is equal to -1. Thus, if $a < m-1$, $w_2(j,m) \geq 0$ except for $a = 2$ and $i = 1$, i.e., for $j = 4$ for which $w_2(j,m) = -1$. Note that this value can be attained only if $2 < m-1$, i.e., if $m \geq 4$.

If $a = m-1$, we have $w_2(j,m) \geq 2^a i - 3a + 1 \geq 2^a - 3a + 1$. Now, since $m \geq 4$, we have $a \geq 3$, hence $w_2(j,m) \geq 8 - 9 + 1 = 0$.

If $a > m$, we have $w_2(j,m) = 2^a i - 2a - m \geq 2^a i - 3a + 1 \geq 2^a - 3a + 1 \geq 0$ for all j, since $m \geq 2$.

Finally, if $a = m$, we have $w_2(j,m) = 2^m i - 3m - 2v(i+1)$. We note that, for all i, we have $v(i+1) \leq i$; thus,

$$w_2(j,m) \geq (2^m - 2)i - 3m \geq 2^m - 3m - 2 \geq 0 \quad \text{for } m \geq 4.$$

This proves Lemma 7, hence Theorem 6. □

Of course, once again this can be generalized to the computation of $v(S_2(h2^m))$ for a fixed odd h.

As already mentioned, the polylogarithms of order k at 2 do not vanish if $k \geq 3$; therefore, the corresponding sums $S_k(2^m)$ have a bounded valuation. Using the same methods, one can prove the following theorem.

Theorem 8: Denote by lg k the base 2 logarithm of k, set $e(k) = \lceil \lg k \rceil$ and $\delta(k) = 1$ if k is a power of 2, and $\delta(k) = 0$, otherwise. Then, for $k \geq 3$, we have $\mathrm{Li}_k(2) \neq 0$, and in fact

$$v(\mathrm{Li}_k(2)) = 2^{3(k)} - ke(k) + \delta(k).$$

More precisely (still for $k \geq 3$), if

$$S_k(N) = \sum_{j=1}^{N} \frac{2^j}{j^k},$$

then

$$v(S_k(N)) = 2^{e(k)} - ke(k) + \delta(k) \text{ for } N \geq 2^{e(k)+\delta(k)}.$$

Proof: It is clear that all the statements of the theorem follow from the last. Assume first that k is not a power of 2. Then, if we set $w_k(j) = v(2^j / j^k)$ and $j = 2^a i$ with i odd, we have $w_k(j) = 2^a i - ka$. For fixed a, this is minimal for $i = 1$. Furthermore, if we set $f(a) = 2^a - ka$, it is clear that f attains its minimum on the integers for $a = e(k)$, and that this minimum is unique if a is not a power of 2. Hence, there is a single term with minimum valuation for $j = 2^{e(k)} \leq N$, by assumption, so $v(S_k(N)) = 2^{e(k)} - ke(k)$, as claimed.

Assume now that a is a power of 2. Then the minimum of f is attained for $a = e(k)$ and for $a = e(k)+1$. The corresponding terms in the sum not only have the same valuation, but are in fact equal, hence the valuation w of their sum is simply 1 more than usual. We now notice that $f(a+1) - f(a) = 2^a - 2^{e(k)}$. Therefore, since we have assumed $k \geq 3$, hence $e(k) \geq 2$, we have $|f(a+1) - f(a)| \geq 2$ for $a \neq e(k)$, so all the other terms have a valuation that is strictly larger than w, so $v(S_k(N)) = w = 2^{e(k)} - ke(k) +$ for $N \geq 2^{e(k)+1}$, as claimed. □

Remark: One can generalize the above results to other primes than $p = 2$, but the results are much less interesting. For example, it is easy to show, using similar methods, that the 3-adic valuation of

$$\sum_{j=1}^{3^m} (2 + (-1)^{j-1}) \frac{3^j}{j}$$

is equal to $3^m + 1$ for all m.

REFERENCES

1. T. Lengyel. "On the Divisibility by 2 of the Stirling Numbers of the Second Kind." *The Fibonacci Quarterly* **32.3** (1994):194-201.
2. T. Lengyel. "Characterizing the 2-adic Order of the Logarithm." *The Fibonacci Quarterly* **32.5** (1994):397-401.
3. K. Yu. "Linear Forms in *p*-acid Logarithms." *Acta Arith.* **53** (1989):107-86,

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