ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to stanley@tiac.net *on the Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-872 ***Proposed by Murray S. Klamkin, University of Alberta, Canada***

Let $r_n = F_{n+1}/F_n$ for $n > 0$. Find a recurrence for r_n^2.

B-873 ***Proposed by Herta T. Freitag, Roanoke, VA***

Prove that 3 is the only positive integer that is both a prime number and of the form $L_{3n} + (-1)^n L_n$.

B-874 ***Proposed by David M. Bloom, Brooklyn College, NY***

Prove that 3 is the only positive integer that is both a Fibonacci number and a Mersenne number. [A Mersenne number is a number of the form $2^a - 1$.]

B-875 ***Proposed by Richard André-Jeannin, Cosnes et Romain, France***

Prove that 3 is the only positive integer that is both a triangular number and a Fermat number. [A triangular number is a number of the form $n(n+1)/2$. A Fermat number is a number of the form $2^a + 1$.]

B-876 ***Proposed by N. Gauthier, Royal Military College of Canada***

Evaluate

$$\sum_{k=1}^{n} \sin\left(\frac{\pi F_{k-1}}{F_k F_{k+1}}\right) \sin\left(\frac{\pi F_{k+2}}{F_k F_{k+1}}\right).$$

B-877 ***Proposed by Indulis Strazdins, Riga Technical University, Latvia***

Evaluate

$$\begin{vmatrix} F_nF_{n+1} & F_{n+1}F_{n+2} & F_{n+2}F_{n+3} & F_{n+3}F_{n+4} \\ F_{n+4}F_{n+5} & F_{n+5}F_{n+6} & F_{n+6}F_{n+7} & F_{n+7}F_{n+8} \\ F_{n+8}F_{n+9} & F_{n+9}F_{n+10} & F_{n+10}F_{n+11} & F_{n+11}F_{n+12} \\ F_{n+12}F_{n+13} & F_{n+13}F_{n+14} & F_{n+14}F_{n+15} & F_{n+15}F_{n+16} \end{vmatrix}.$$

SOLUTIONS

The Right Angle to Success

B-854 ***Proposed by Paul S. Bruckman, Edmonds, WA***
(Vol. 36, no. 3, August 1998)

Simplify

$$3\arctan(\alpha^{-1}) - \arctan(\alpha^{-5}).$$

Solution by L. A. G. Dresel, Reading, England

Let $\theta = \arctan(\alpha^{-1})$, so that $\tan\theta = \alpha^{-1}$. Using the formula

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y},$$

we find that

$$\tan 2\theta = 2\alpha^{-1}/(1-\alpha^{-2}) = 2\alpha/(\alpha^2-1) = 2\alpha/\alpha = 2,$$

and

$$\tan 3\theta = (2+\alpha^{-1})/(1-2\alpha^{-1}) = (2-\beta)/(1+2\beta) = (1+\alpha)/(\beta^2+\beta) = \alpha^2/\beta^3 = -\alpha^5.$$

Hence, $3\arctan(\alpha^{-1}) = \pi - \arctan(\alpha^5)$, and since $\arctan(\alpha^{-5}) + \arctan(\alpha^5) = \pi/2$, we have

$$3\arctan(\alpha^{-1}) - \arctan(\alpha^{-5}) = \pi/2.$$

Solutions also received by Richard André-Jeannin, Charles K. Cook, Steve Edwards, Russell Jay Hendel, Walther Janous, Murray S. Klamkin, Angel Plaza & Miguel A. Padrón, Maitland A. Rose, Jaroslav Seibert, H.-J. Seiffert, Indulis Strazdins, and the proposer.

Recurrence for a Ratio

B-855 ***Proposed by the editor***
(Vol. 36, no. 3, August 1998)

Let $r_n = F_{n+1}/F_n$ for $n > 0$. Find a recurrence for r_n.

Solution by Steve Edwards, Southern Polytechnic State University, Marietta, GA

$$r_n = \frac{F_{n+1}}{F_n} = \frac{F_n + F_{n-1}}{F_n} = 1 + \frac{F_{n-1}}{F_n} = 1 + \frac{1}{r_{n-1}} \quad \text{for } n > 1.$$

Generalization by Murray S. Klamkin, University of Alberta, Canada: More generally, we determine a recurrence for $r_n = G_{n+1}/G_n$, where $G_{n+1} = aG_n + bG_{n-1}$ by simply dividing the latter recurrence by G_n to give

$$r_n = a + b/r_{n-1}.$$

Klamkin gave generalizations to third-order recurrences as well as several other generalizations, one of which we present to the readers as problem B-872 in this issue.

Solutions also received by Richard André-Jeannin, Paul S. Bruckman, Charles K. Cook, Mario DeNobili, Leonard A. G. Dresel, Herta T. Freitag, Pentti Haukkanen, Russell Jay Hendel, Walther Janous, Daina Krigens, Angel Plaza & Miguel A. Padrón, Jaroslav Seibert, H.-J. Seiffert, Indulis Strazdins, and the proposer.

Weak Inequality

B-856 ***Proposed by Zdravko F. Starc, Vršac, Yugoslavia (Vol. 36, no. 3, August 1998)***

If n is a positive integer, prove that

$$L_1\sqrt{F_1}+L_2\sqrt{F_2}+L_3\sqrt{F_3}+\cdots+L_n\sqrt{F_n}<8F_n^2+4F_n.$$

Solution 1 by Richard André-Jeannin, Cosnes et Romain, France

We see that

$$\begin{aligned}L_1\sqrt{F_1}+L_2\sqrt{F_2}+L_3\sqrt{F_3}+\cdots+L_n\sqrt{F_n} &\le L_1F_1+L_2F_2+\cdots+L_nF_n\\ &=F_2+F_4+\cdots+F_{2n}=F_{2n+1}-1<F_{2n+1}\\ &=F_n^2+F_{n+1}^2<F_n^2+(2F_n)^2=5F_n^2<8F_n^2+4F_n.\end{aligned}$$

Solution 2 by L. A. G. Dresel, Reading, England

We shall prove the much stronger result

$$L_1\sqrt{F_1}+L_2\sqrt{F_2}+L_3\sqrt{F_3}+\cdots+L_n\sqrt{F_n}<4.35F_n^{3/2}.$$

Let $\gamma=\beta/\alpha=-\alpha^2$ and $\delta=\sqrt{5}$. Then $L_k=\alpha^k(1+\gamma^k)$, $F_k=\alpha^k(1-\gamma^k)/\delta$, $\sqrt{F_k}<\alpha^{k/2}(1-\gamma^k/2)/\sqrt{\delta}$, and $L_k\sqrt{F_k}<\alpha^{3k/2}(1+\gamma^k/2)/\sqrt{\delta}$. Summing for $1\le k\le n$, we have two geometric progressions, giving

$$\begin{aligned}\Sigma L_k\sqrt{F_k}&<(\alpha^{3(n+1)/2}-\alpha^{3/2})/(\alpha^{3/2}-1)\sqrt{\delta}-\tfrac{1}{2}(\alpha^{-1/2}-(-1)^n\alpha^{-(n+1)/2})/(1+\alpha^{-1/2})\sqrt{\delta}\\ &<(\alpha^{3n/2}-1)/(1-\alpha^{-3/2})\sqrt{\delta}.\end{aligned}$$

Now

$$\begin{aligned}F_n^{3/2}&=\alpha^{3n/2}(1-\gamma^n)^{3/2}/\delta^{3/2}>\alpha^{3n/2}(1-3\gamma^n/2)/\delta^{3/2}\\ &>(\alpha^{3n/2}-3/2\alpha)/\delta^{3/2}>(\alpha^{3n/2}-1)/\delta^{3/2}.\end{aligned}$$

Hence,

$$\Sigma L_k\sqrt{F_k}<cF_n^{3/2},$$

where $c=\sqrt{5}/(1-\alpha^{-3/2})=4.34921\ldots<4.35$.

All solvers strengthened the proposed equality. Upper bounds found were:

Jaroslav Seibert:	$7F_n^2-2F_n$
H.-J. Seiffert:	$5F_n^2$
Walther Janous:	$8F_n^{3/2}$
Paul S. Bruckman:	$2.078\sqrt{F_{3n}}$

Linear Number of Digits

B-857 ***Proposed by the editor***
(Vol. 36, no. 3, August 1998)

Find a sequence of integers $\langle w_n \rangle$ satisfying a recurrence of the form $w_{n+2} = Pw_{n+1} - Qw_n$ for $n \ge 0$ such that, for all $n > 0$, w_n has precisely n digits (in base 10).

Solution by Richard André-Jeannin, Cosnes et Romain, France

The sequence $w_n = 10^n - 1$ has n digits in base 10 and satisfies the recurrence:

$$w_n = 11w_{n-1} - 10w_{n-2}.$$

Solutions also received by Paul S. Bruckman, Aloysius Dorp, Leonard A. G. Dresel, Gerald A. Heuer, Walther Janous, H.-J. Seiffert, and the proposer.

Calculating Convolutions

B-858 ***Proposed by Wolfdieter Lang, Universität Karlsruhe, Germany***
(Vol. 36, no. 3, August 1998)

(a) Find an explicit formula for $\sum_{k=0}^{n} kF_{n-k}$ which is the convolution of the sequence $\langle n \rangle$ and the sequence $\langle F_n \rangle$.

(b) Find explicit formulas for other interesting convolutions.

(The convolution of the sequence $\langle a_n \rangle$ and $\langle b_n \rangle$ is the sum $\sum_{k=0}^{n} a_k b_{n-k}$.)

Solution to (a) by Steve Edwards, Southern Polytechnic State Univ., Marietta, GA

We show that

$$\sum_{k=0}^{n} kG_{n-k} = G_{n+3} - [(n+2)G_1 + G_0]$$

for any generalized Fibonacci sequence $\langle G_n \rangle$, and this gives as a special case the sum in (a), which sums to $F_{n+3} - (n+3)$.

Proof by induction: For $n = 0$, $0G_0 = 0 = G_3 - (G_2 + G_1) = G_3 - (2G_1 + G_0)$. For $n = m+1$,

$$\begin{aligned}\sum_{k=0}^{m+1} kG_{(m+1)-k} &= \sum_{j=0}^{m} (j+1)G_{m-j} = \sum_{j=0}^{m} jG_{m-j} + \sum_{j=0}^{m} G_{m-j} \\ &= G_{m+3} - [(m+2)G_1 + G_0] + [G_{m+2} - G_1] \quad \text{(by a variation of (33) in [1])} \\ &= G_{m+4} - [(m+3)G_1 + G_0].\end{aligned}$$

Reference

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications.* Chichester: Ellis Horwood Ltd., 1989.

Solution to (b) by H.-J. Seiffert, Berlin, Germany

Let $F_n(x)$ denote the Fibonacci polynomial, defined by $F_0(x) = 0$, $F_1(x) = 1$, and $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$ for $n \ge 0$. Then we have

$$\sum_{k=0}^{n} F_k(x)F_{n-k}(y) = \frac{F_n(x) - F_n(y)}{x - y}.$$

Several solvers found the convolution of $\langle n^2\rangle$ *and* $\langle F_n\rangle$ *to be* $F_{n+6}-(n^2+4n+8)$. *Dresel found the convolution of* $\langle n\rangle$ *and* $\langle L_n\rangle$ *to be* $L_{n+3}-(n+4)$.

Solutions also received by Richard André-Jeannin, Paul S. Bruckman, Leonard A. G. Dresel, Pentti Haukkanen, Walther Janous, Hans Kappus, Murray S. Klamkin, Carl Libis, Jaroslav Seibert, Indulis Strazdins, and the proposer.

Fun Determinant

B-859 ***Proposed by Kenneth B. Davenport, Pittsburgh, PA (Vol. 36, no. 3, August 1998)***

Simplify

$$\begin{vmatrix} F_nF_{n+1} & F_{n+1}F_{n+2} & F_{n+2}F_{n+3} \\ F_{n+3}F_{n+4} & F_{n+4}F_{n+5} & F_{n+5}F_{n+6} \\ F_{n+6}F_{n+7} & F_{n+7}F_{n+8} & F_{n+8}F_{n+9} \end{vmatrix}.$$

Solution by Russell Hendel, Philadelphia, PA

The determinant's value is $32(-1)^n$.

It is easy to verify this for the seven values $n=-3,-2,-1,0,1,2,3$. The result now follows for all n by Dresel's Verification Theorem [1], since the determinant is a homogeneous algebraic form of degree 6.

Reference

1. L. A. G. Dresel. "Transformations of Fibonacci-Lucas Identities." In *Applications of Fibonacci Numbers* **5**:169-84. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1996.

Seiffert found that

$$\begin{vmatrix} F_nF_{n+1} & F_{n+1}F_{n+2} & F_{n+2}F_{n+3} \\ F_{n+p}F_{n+p+1} & F_{n+p+1}F_{n+p+2} & F_{n+p+2}F_{n+p+3} \\ F_{n+q}F_{n+q+1} & F_{n+q+1}F_{n+q+2} & F_{n+q+2}F_{n+q+3} \end{vmatrix} = (-1)^{n+p-1}F_pF_qF_{q-p}.$$

For a related problem, see problem B-877 in this issue.

Solutions also received by Richard André-Jeannin, Paul S. Bruckman, Leonard A. G. Dresel, Walther Janous, Carl Libis, Stanley Rabinowitz, Jaroslav Seibert, H.-J. Seiffert, Indulis Strazdins, and the proposer.

Addenda. We wish to belatedly acknowledge solutions from the following solvers:

Murray S. Klamkin—B-848, 849, 850, 851
Harris Kwong—B-831, 832
A. J. Stam—B-853

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