DIVISION OF FIBONACCI NUMBERS BY k

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1. AIM AND SCOPE OF THE PAPER

Notation

(i) F_n and L_n denote the n^{th} Fibonacci and Lucas number, respectively.

(ii) r and s denote the residue of the subscript n [e.g., see (1.1)] modulo 2 and 4, respectively.

(iii) The symbol $\lfloor\cdot\rfloor$ denotes the greatest integer function.

(iv) $Z(k)$ denotes the so-called *entry point* of k in the Fibonacci sequence, that is, the smallest subscript m for which F_m is divisible by k.

(v) $P(k)$ denotes the repetition period of the Fibonacci sequence reduced modulo k.

The aim of this paper is to extend some results concerning the *Zeckendorf decomposition* (ZD, in brief) [6] of integers of the form F_n / d that have been established in [4]. More precisely, we shall determine the ZD of the integers of the form

$$F_{Z(k)n} / k \quad (n = 1, 2, 3, \dots) \tag{1.1}$$

for certain values of the integer $k \geq 2$. The integrality of the numbers (1.1) is ensured by the definition of $Z(k)$ and by the well-known property $F_{nm} \equiv 0 \pmod{F_m}$.

This kind of study is, obviously, "endless" so that the choice of the values of k in (1.1) posed some problems for us. After some thinking, we decided to consider all values of $k \leq 11$ and the prime values of $k \leq 23$ (Section 2). More interesting results emerge from the ZD of (1.1) for special values of k, as shown in Sections 4, 5, and 6. The numerical values of $Z(k)$ have been taken from the list available in [1, pp. 33-41].

All of the results have been established by proving conjectures based on the behavior which became apparent through a study of early cases of n, once k was chosen. Conjecturing these results is, in most cases, a laborious task involving the use of a software package that can handle large-subscripted Fibonacci numbers, and a computer program for the ZD of large integers. On the other hand, once the conjectures are made, the proofs are not difficult but, in general, they are very lengthy and tedious so that giving them for all the results is unreasonable. In fact, only the (partial) proof of (2.4) and the complete proof of (5.4) are given (resp. Sections 3 and 5), just to show the technique we used. The main mathematical tools used in the proofs are the identities (1.4)-(1.6) of [3].

Note that $F_1 = 1$ and $F_2 = 1$ are used indifferently in the ZDs. Moreover, as usual, we assume that a sum vanishes whenever its upper range indicator is less than the lower one.

2. RESULTS

$$F_{3n}/2=\sum_{j=1}^{n}F_{3j-2}. \tag{2.1}$$

$$F_{4n}/3=rF_2+\sum_{j=1}^{\lfloor n/2\rfloor}(F_{4(2j+r)-5}+F_{4(2j+r)-3}). \tag{2.2}$$

$$F_{6n}/4=\sum_{j=1}^{n}F_{6j-3}. \tag{2.3}$$

$$F_{5n}/5=x(n)+\sum_{j=1}^{\lfloor n/4\rfloor}\left(F_{5(4j+s)-17}+F_{5(4j+s)-14}+F_{5(4j+s)-4}+\sum_{i=1}^{3}F_{5(4j+s-1)-2i}\right), \tag{2.4}$$

where

$$x(n)=\begin{cases}0, & \text{if } s=0,\\ F_2, & \text{if } s=1,\\ F_4+F_6, & \text{if } s=2,\\ F_2+F_4+F_6+F_8+F_{11}, & \text{if } s=3.\end{cases} \tag{2.4$'$}$$

$$F_{12n}/6=r(F_4+F_8)+\sum_{j=1}^{\lfloor n/2\rfloor}\left(F_{12(2j+r)-21}+F_{12(2j+r)-18}+F_{12(2j+r)-16}+F_{12(2j+r)-4}+\sum_{i=1}^{3}F_{12(2j+r)-2i-7}\right). \tag{2.5}$$

$$F_{8n}/7=rF_4+\sum_{j=1}^{\lfloor n/2\rfloor}\sum_{i=1}^{4}F_{8(2j+r)-2i-3}. \tag{2.6}$$

$$F_{6n}/8=rF_2+\sum_{j=1}^{\lfloor n/2\rfloor}(F_{6(2j+r)-7}+F_{6(2j+r)-5}). \tag{2.7}$$

$$F_{12n}/9=r(F_4+F_7)+\sum_{j=1}^{\lfloor n/2\rfloor}\left(F_{12(2j+r)-19}+F_{12(2j+r)-5}+\sum_{i=1}^{4}F_{12(2j+r)-2i-7}\right). \tag{2.8}$$

The complete ZD of $F_{15n}/10$ is extremely cumbersome and unpleasant. We confine ourselves to showing it only for $s=0$.

$$F_{15n}/10=\sum_{j=1}^{n/4}\left(F_{60j-55}+F_{60j-51}+F_{60j-48}+F_{60j-42}+F_{60j-40}+F_{60j-37}+F_{60j-23}+F_{60j-21}+F_{60j-16}+F_{60j-14}+F_{60j-10}+F_{60j-5}+\sum_{i=1}^{4}F_{60j-2i-25}\right). \tag{2.9}$$

$$F_{10n}/11=\sum_{j=1}^{n}F_{10j-5}. \tag{2.10}$$

$$F_{7n}/13=x(n)+\sum_{j=1}^{\lfloor n/4\rfloor}\left(F_{7(4j+s)-23}+F_{7(4j+s)-20}+F_{7(4j+s)-6}+\sum_{i=1}^{5}F_{7(4j+s-1)-2i}\right), \tag{2.11}$$

where

$$x(n)=\begin{cases}0, & \text{if } s=0,\\ F_2, & \text{if } s=1,\\ F_6+F_8, & \text{if } s=2,\\ F_2+F_4+F_6+F_8+F_{10}+F_{12}+F_{15}, & \text{if } s=3.\end{cases} \tag{2.11'}$$

$$F_{9n}/17=x(n)+\sum_{j=1}^{\lfloor n/4\rfloor}\left(F_{9(4j+s)-31}+F_{9(4j+s)-6}+\sum_{i=1}^{3}F_{9(4j+s)-2i-22}+\sum_{i=1}^{5}F_{9(4j+s)-2i-11}\right), \tag{2.12}$$

where

$$x(n)=\begin{cases}0, & \text{if } s=0,\\ F_3, & \text{if } s=1,\\ F_6+F_{12}, & \text{if } s=2,\\ F_4+F_6+F_8+F_{10}+F_{12}+F_{14}+F_{21}, & \text{if } s=3.\end{cases} \tag{2.12'}$$

$$F_{18n}/19=\sum_{j=1}^{n}\sum_{i=1}^{3}F_{18j-2i-5}. \tag{2.13}$$

$$\begin{aligned}F_{24n}/23=r(F_6+F_9+F_{14}+F_{17})+\sum_{j=1}^{\lfloor n/2\rfloor}\Bigg(&F_{24(2j+r)-41}+F_{24(2j+r)-39}+F_{24(2j+r)-36}\\ &+F_{24(2j+r)-34}+F_{24(2j+r)-15}+F_{24(2j+r)-10}+F_{24(2j+r)-7}+\sum_{i=1}^{6}F_{24(2j+r)-2i-17}\Bigg).\end{aligned} \tag{2.14}$$

3. A PROOF

Proof of (2.4) (for $s=1$)

Use (1.4)-(1.6) of [3] and (2.4′) to rewrite the right-hand side of (2.4) as

$$\begin{aligned}
&F_2+\sum_{j=1}^{(n-1)/4}F_{20j-12}+\sum_{j=1}^{(n-1)/4}(F_{20j-9}+F_{20j+1})+\sum_{j=1}^{(n-1)/4}\sum_{i=1}^{3}F_{20j-2i}\\
&=F_2+\sum_{j=1}^{(n-1)/4}F_{20j-12}+5\sum_{j=1}^{(n-1)/4}L_{20j-4}+4\sum_{j=1}^{(n-1)/4}F_{20j-4}\\
&=F_2+\frac{F_{5n+3}-F_{5n-17}-165}{L_{20}-2}+5\frac{L_{5n+11}-L_{5n-9}-2200}{L_{20}-2}+4\frac{F_{5n+11}-F_{5n-9}-990}{L_{20}-2}\\
&=F_2+\frac{55L_{5n-7}-165}{15125}+5\frac{275F_{5n+1}-2200}{15125}+4\frac{55L_{5n+1}-990}{15125}\\
&=\frac{L_{5n-7}+25F_{5n+1}+4L_{5n+1}}{275}=\frac{L_{5n-7}+5L_{5n}+5L_{5n+2}+4L_{5n+1}}{275}\quad(\text{from } I_9 \text{ of } [5])\\
&=\frac{L_{5n-7}+10L_{5n+2}-L_{5n+1}}{275}=\frac{10L_{5n+2}-15F_{5n-3}}{275}=\frac{2L_{5n+2}-3F_{5n-3}}{55}
\end{aligned}$$

$$=\frac{2F_{5n+3}+2F_{5n+1}-3F_{5n-3}}{55}=\frac{2(F_{5n+3}-F_{5n-3})+2F_{5n+1}-F_{5n-3}}{55}$$

$$=\frac{8F_{5n}+2F_{5n+1}-F_{5n-3}}{55}=\frac{8F_{5n}+3F_{5n}}{55}=\frac{F_{5n}}{5}.\quad \text{Q.E.D.}$$

We do not exclude the possibility that a shorter proof can be given.

4. RESULTS FOR SOME SPECIAL VALUES OF k

From the results presented in Section 2, we noted that the ZD of $F_{Z(k)n}/k$ is independent of the residue of n modulo 2 or 4 whenever

$$Z(k)=P(k), \tag{4.1}$$

so that its expression assumes a rather simple form [cf. (2.1), (2.3), (2.10), and (2.13)]. We were not able to find conditions for (4.1) to hold for k arbitrary. On the other hand, it is known (see (19) of [2]) that, if p is a prime, then

$$Z(p)=P(p) \text{ iff } F_{Z(p)-1}\equiv 1 \pmod{p}. \tag{4.2}$$

A list of the first few values of k satisfying (4.1) and their $Z(k)$ is available in [1, p. 33]. In the following, we show the ZD of $F_{Z(k)n}/k$ for all such k greater than 19 [cf. (2.13)] and not exceeding 71:

$$F_{30n}/22=\sum_{j=1}^{n}\left[F_{30j-23}+\sum_{i=1}^{3}(F_{30j-3i-12}+F_{30j-3i-4})\right]; \tag{4.3}$$

$$F_{14n}/29=\sum_{j=1}^{n}F_{14j-7}; \tag{4.4}$$

$$F_{30n}/31=\sum_{j=1}^{n}\left(F_{30j-15}+F_{30j-10}+\sum_{i=1}^{4}F_{30j-5i-3}\right); \tag{4.5}$$

$$F_{18n}/38=\sum_{j=1}^{n}(F_{18j-11}+F_{18j-8}); \tag{4.6}$$

$$F_{30n}/44=\sum_{j=1}^{n}(F_{30j-23}+F_{30j-20}+F_{30j-18}+F_{30j-14}+F_{30j-8}); \tag{4.7}$$

$$F_{42n}/58=\sum_{j=1}^{n}\left(F_{42j-33}+F_{42j-20}+\sum_{i=1}^{3}F_{42j-3i-22}+\sum_{i=1}^{4}F_{42j-3i-6}\right); \tag{4.8}$$

$$F_{58n}/59=\sum_{j=1}^{n}\Bigg(F_{58j-47}+F_{58j-44}+F_{58j-26}+F_{58j-16}+F_{58j-12}+F_{58j-9} +\sum_{i=1}^{3}F_{58j-14i-7}+\sum_{i=1}^{5}F_{58j-5i-13}\Bigg); \tag{4.9}$$

$$F_{30n}/62=\sum_{j=1}^{n}\left(F_{30j-21}+F_{30j-9}+\sum_{i=1}^{3}F_{30j-2i-11}\right); \tag{4.10}$$

$$F_{70n}/71=\sum_{j=1}^{n}\Bigg[F_{70j-61}+F_{70j-48}+F_{70j-36}+F_{70j-23}+F_{70j-15}+F_{70j-9} + \sum_{i=1}^{3}(F_{70j-12i-17}+F_{70j-5i-40})+\sum_{i=1}^{4}F_{70j-7i-11}\Bigg]. \tag{4.11}$$

5. RESULTS FOR *VERY* SPECIAL VALUES OF k

Inspection of the results established in Sections 2 and 4 shows that, for $k=2$, 4, 11, and 29, the ZD of $F_{Z(k)n}/k$ is constituted by exactly n addends. If we disregard the value 2, it is quite natural to conjecture that the ZD of $F_{Z(L_{2h+1})n}/L_{2h+1}$ ($h=1,2,3,\ldots$) has n addends.

Question 1. What is the value of $Z(L_{2h+1})$?

Theorem IV of [5, p. 40], which is credited to L. Carlitz, immediately gives the answer:

$$Z(L_{2h+1})=4h+2. \tag{5.1}$$

In fact, we state the following proposition.

Proposition 1: For $h=1,2,3,\ldots$, we have

$$\frac{F_{Z(L_{2h+1})n}}{L_{2h+1}}=\frac{F_{(4h+2)n}}{L_{2h+1}}=\sum_{j=1}^{n}F_{(4h+2)j-2h-1}. \tag{5.2}$$

The proof of Proposition 1 can be obtained simply by letting $n=2h+1$ and $k=n$ in (2.2) of [3].

Question 2. Apart from the particular case $k=2$ [see (2.1)], do there exist other values of k, that are not odd-subscripted Lucas numbers, for which the ZD of $F_{Z(k)n}/k$ is constituted by exactly n addends?

A computer search showed that none of them exists for $k\le 1000$. This search has not been completely useless as it allowed us to discover a misprint in Brousseau's table of entry points [1, pp. 33-41] where it is reported that $Z(961)=839$; the correct value is 930.

The ZD of $F_{18n}/38$ has exactly $2n$ addends [see (4.6)]. The only further value of $k\le 1000$ for which such a decomposition occurs is $k=682$. Namely, we have

$$F_{30n}/682=\sum_{j=1}^{n}(F_{30j-17}+F_{30j-14}). \tag{5.3}$$

The decompositions (4.6) and (5.3) led us to discover the following result, the proof of which is appended below.

Proposition 2: For $h=1,2,3,\ldots$, we have

$$\frac{F_{Z(L_{6h+3}/2)n}}{L_{6h+3}/2}=\frac{F_{(12h+6)n}}{L_{6h+3}/2}=\sum_{j=1}^{n}(F_{(12h+6)j-6h-5}+F_{(12h+6)j-6h-2}). \tag{5.4}$$

Remark: Conjecturing this result has been very laborious but, once the conjecture has been made, its proof is quite easy. Observe that expression (5.4) works for $h=0$ as well, but $12\cdot 0+6=6$ is not the entry point of $L_{6\cdot 0+3}/2=2$.

Proof of Proposition 2: First, observe that, for $h \geq 1$,

$$L_{6h+3} = 4v \quad \text{with} \quad v = 1 + \sum_{r=1}^{h} L_{6r} \quad (v > 1 \text{ odd}), \tag{5.5}$$

and use (5.5) and the basic properties of $Z(k)$ [2, p. 9] to establish that

$$Z(L_{6h+3}/2) = Z(L_{6h+3}) \quad (h \geq 1). \tag{5.6}$$

Then, use Theorem IV of [5, p. 40] to write

$$Z(L_{6h+3}) = 12h + 6. \tag{5.7}$$

Finally, rewrite the right-hand side of (5.4) as

$$2\sum_{j=1}^{n} F_{(12h+6)j-6h-3} = \frac{2[F_{(12h+6)n+6h+3} - F_{(12h+6)n-6h-3}]}{L_{12h+6} - 2} \quad \text{(from (1.4) of [3])}$$

$$= \frac{2F_{(12h+6)n}L_{6h+3}}{L_{12h+6} - 2} = \frac{2F_{(12h+6)n}L_{6h+3}}{L^2_{6h+3}} \quad \text{(from (1.5) of [3] and } I_{18} \text{ of [5])}$$

$$= \frac{F_{(12h+6)n}}{L_{6h+3}/2} = \frac{F_{Z(L_{6h+3}/2)n}}{L_{6h+3}/2} \quad \text{[from (5.7) and (5.6)]. Q.E.D.}$$

6. CONCLUDING REMARKS

The characterization of classes of integers k for which the ZD of $F_{Z(k)n}/k$ is constituted by n q-tuples of Fibonacci numbers whose subscripts are in arithmetical progression seems to be an interesting subject of study, and might be the aim of a future investigation. Propositions 1 and 2 give a solution to this problem for $q = 1$ and 2, respectively. For $q = 3$, we state the following proposition [cf. (2.13)].

Proposition 3: For $h = 1, 2, 3, \ldots$, we have

$$\frac{F_{Z(L_{6h+3}/4)n}}{L_{6h+3}/4} = \frac{F_{(12h+6)n}}{L_{6h+3}/4} = \sum_{j=1}^{n}\sum_{i=1}^{3} F_{(12h+6)j-2i-6h+1}. \tag{6.1}$$

The decompositions

$$\frac{F_{30n}}{L_{10}+1} = \sum_{j=1}^{n}\sum_{i=1}^{5} F_{30j-2i-9} \quad \text{and} \quad \frac{F_{42n}}{L_{14}+1} = \sum_{j=1}^{n}\sum_{i=1}^{7} F_{42j-2i-13} \tag{6.2}$$

seem to be a good starting point for the above-mentioned investigation. In fact, they led us to claim that

$$\frac{F_{Z(L_{2q}+1)n}}{L_{2q}+1} = \frac{F_{6qn}}{L_{2q}+1} = \sum_{j=1}^{n}\sum_{i=1}^{q} F_{6qj-2i-2q+1} \quad (q \text{ odd}). \tag{6.3}$$

Observe that both letting $h = 1$ in (6.1) and letting $q = 3$ in (6.3), yield (2.13). The proofs of (6.1) and (6.3) are left as an exercise for the interested reader. Note that the proof of (6.3) involves the use of the identity $L_{3q}/L_q = L_{2q} + 1$ (see (2.5) of [3]).

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REFERENCES

1. Bro. A. Brousseau. *Fibonacci and Related Number Theoretic Tables.* Santa Clara, CA: The Fibonacci Association, 1972.
2. D. C. Fielder & P. S. Bruckman. *Fibonacci Entry Points and Periods for Primes 100,003 through 415,993.* Santa Clara, CA: The Fibonacci Association, 1995.
3. P. Filipponi & H. T. Freitag. "The Zeckendorf Decomposition of Certain Classes of Integers." In *Applications of Fibonacci Numbers* **6:**123-35. Dordrecht: Kluwer, 1996.
4. H. T. Freitag & P. Filipponi. "On the Representation of Integral Sequences $\{F_n / d\}$ and $\{L_n / d\}$ as Sums of Fibonacci Numbers and as Sums of Lucas Numbers." In *Applications of Fibonacci Numbers* **2:**97-112. Dordrecht: Kluwer, 1988.
5. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Boston: Houghton-Mifflin, 1969; rpt. Santa Clara, CA: The Fibonacci Association, 1979.
6. E. Zeckendorf. "Répresentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas." *Bull. Soc. Roy. Sci. Liège* **41** (1972):179-82.

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