NOTES ON RECIPROCAL SERIES RELATED TO FIBONACCI AND LUCAS NUMBERS

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1. INTRODUCTION

As usual, the Fibonacci and Lucas numbers are defined by

$$F_n = \frac{\alpha^n - (-1)^n \alpha^{-n}}{\sqrt{5}}, \quad L_n = \alpha^n + (-1)^n \alpha^{-n},$$

where $\alpha = (\sqrt{5}+1)/2$. Sums of the form $\sum 1/(F_{an+b}+c)$ or $\sum 1/(L_{an+b}+c)$ have been computed in many publications for certain values of a, b, and c (see, for instance, [2]-[5]). For example: Backstrom [3] obtained

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n}+3} = \frac{2\sqrt{5}+1}{10}, \quad \sum_{n=0}^{\infty} \frac{1}{F_{2n+1}+1} = \frac{\sqrt{5}}{2};$$

André-Jeannin [2] proved that

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n}+\sqrt{5}} = \frac{1}{\alpha}, \quad \sum_{n=0}^{\infty} \frac{1}{F_{2n+1}+3/\sqrt{5}} = 1,$$

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}+2/\sqrt{5}} \approx \frac{\sqrt{5}}{4\log\alpha} - \frac{\sqrt{5}\pi^2}{(\log\alpha)^2(e^{\pi^2(\log\alpha)^{-1}}+2)};$$

and Almkvist [1] also gave an estimate of another series, i.e.,

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n}+2} \approx \frac{1}{8} + \frac{1}{4\log\alpha} + \frac{\pi^2}{(\log\alpha)^2(e^{\pi^2(\log\alpha)^{-1}}-2)}.$$

In this paper, we continue this work and obtain some new results of similar kinds.

In Section 2, some identities related to Fibonacci and Lucas numbers, which may be compared with the ones of [2] and [3], are established. In Section 3, following Almkvist's method, we express the series $\sum_{n=0}^{\infty} 1/(F_{2n+1}-2/\sqrt{5})$ and $\sum_{n=1}^{\infty} 1/(L_{2n}-2)$ in terms of the theta functions and give their estimates.

2. MAIN RESULTS

The following lemma will be used later.

Lemma: Let q be a real number with $|q|>1$, s and a be positive integers, and b be a nonnegative integer. Then one has that

$$\sum_{n=0}^{\infty} \frac{1}{q^{2an+b}+q^{-2an-b}-(q^{as}+q^{-as})} = \frac{1}{q^{-as}-q^{as}} \sum_{n=0}^{s-1} \frac{1}{1-q^{2an+b-as}} \tag{1}$$

$(b \neq as,\ 2a \nmid (as-b))$.

Proof: One can readily verify that

$$\frac{1}{q^{2n+b}+q^{-2an-b}-(q^{as}+q^{-as})}=\frac{1}{q^{as}-q^{-as}}\left(\frac{1}{1-q^{2an+b+as}}-\frac{1}{1-q^{2an+b-as}}\right)$$

holds for $n>s$. Hence, by the telescoping effect, one has that

$$\sum_{n=0}^{N}\frac{1}{q^{2an+b}+q^{-2an-b}-(q^{as}+q^{-as})}=\frac{1}{q^{as}-q^{-as}}\left(\sum_{n=N-s+1}^{N}\frac{1}{1-q^{2an+b+as}}-\sum_{n=0}^{s-1}\frac{1}{1-q^{2an+b-as}}\right)$$

for all $N>s$. Letting $N\to+\infty$, we obtain the equality (1). □

From the Lemma, some reciprocal series related to Fibonacci and Lucas numbers can be computed.

Theorem 1: Assume that a and b are integers with $a\geq 1$ and $b\geq 0$. Then

$$\sum_{n=0}^{\infty}\frac{1}{F_{2an+b}-L_a/\sqrt{5}}=\frac{1}{F_a(\alpha^{b-a}-1)}\quad (a \text{ even, } b \text{ odd}), \tag{2}$$

$$\sum_{n=0}^{\infty}\frac{1}{L_{2an+b}-\sqrt{5}F_a}=\frac{1}{L_a(\alpha^{b-a}-1)}\quad (a \text{ odd, } b \text{ even}), \tag{3}$$

$$\sum_{n=0}^{\infty}\frac{1}{L_{2an+b}-L_a}=\frac{1}{\sqrt{5}F_a(\alpha^{b-a}-1)}\quad (a,\ b \text{ even, } a\neq b, \text{ and } 2a\nmid(a-b)), \tag{4}$$

and

$$\sum_{n=0}^{\infty}\frac{1}{F_{2an+b}-F_a}=\frac{\sqrt{5}}{L_a(\alpha^{b-a}-1)}\quad (a,\ b \text{ odd, } a\neq b, \text{ and } 2a\nmid(a-b)). \tag{5}$$

Proof: Set $q=\alpha$. It follows from (1) that

$$\sum_{n=0}^{\infty}\frac{1}{\alpha^{2an+b}+\alpha^{-2an-b}-(\alpha^{as}+\alpha^{-as})}=\frac{1}{\alpha^{-as}-\alpha^{as}}\sum_{n=0}^{s-1}\frac{1}{1-\alpha^{2an-as+b}}. \tag{6}$$

Examine different cases according to the values of a, b, and s. If a is even, b is odd, and $s=1$, then we have (2) from (6). If a is odd, b is even, and $s=1$ in (6), then (3) holds. On the other hand, assume that $s=1$, $2a\nmid(a-b)$, and $a\neq b$ in (6), then we have the equalities (4) and (5) if both a and b are even or odd, respectively. □

Theorem 2: Suppose that s is a positive integer. Then

$$\sum_{n=0}^{\infty}\frac{1}{L_{2n}-\sqrt{5}F_s}=\frac{1}{L_s}\left(\frac{1-s}{2}+\frac{1}{\alpha^{-s}-1}\right)\quad (s \text{ odd}) \tag{7}$$

and

$$\sum_{n=0}^{\infty}\frac{1}{F_{2n+1}-L_s/\sqrt{5}}=\frac{s}{2F_s}\quad (s \text{ even}). \tag{8}$$

Proof: Letting $a=1$ and $b=0$ in (6), we have

$$\sum_{n=0}^{\infty}\frac{1}{L_{2n}-\sqrt{5}F_s}=-\frac{1}{L_s}\sum_{n=0}^{s-1}\frac{1}{1-\alpha^{2n-s}}\quad (s \text{ odd}).$$

Due to

$$\sum_{n=0}^{2m} \frac{1}{1-\alpha^{2n-2m-1}} = \frac{1}{1-\alpha^{-2m-1}} + \sum_{n=1}^{m}\left(\frac{1}{1-\alpha^{-2n+1}} + \frac{1}{1-\alpha^{-2n-1}}\right) = \frac{1}{1-\alpha^{-2m-1}} + m,$$

we obtain the equality (7). On the other hand, if $a = b = 1$ in (6), then

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1} - L_s/\sqrt{5}} = -\frac{1}{F_s}\sum_{n=0}^{s-1} \frac{1}{1-\alpha^{2n+1-s}} \quad (s \text{ even}).$$

Noticing that

$$\sum_{n=0}^{2m-1} \frac{1}{1-\alpha^{2n+1-2m}} = \sum_{n=1}^{m}\left(\frac{1}{1-\alpha^{-2n+1}} + \frac{1}{1-\alpha^{2n-1}}\right) = m,$$

we have the equality (8). □

Remark: Consider the recurrence relation $W_n = pW_{n-1} + W_{n-2}$, $n \geq 2$, $p > 0$, and the solutions

$$U_n = \frac{\alpha^n - (-1)^n \alpha^{-n}}{\sqrt{\Delta}}, \quad V_n = \alpha^n + (-1)^n \alpha^{-n},$$

where $\Delta = p^2 + 4$, $\alpha = (p + \sqrt{\Delta})/2 > 1$. $\{U_n\}$ and $\{V_n\}$ are the generalizations of $\{F_n\}$ and $\{L_n\}$. Clearly, the conclusions of Theorems 1 and 2 can be generalized to the case in which F_n, L_n, and $\sqrt{5}$ are replaced by U_n, V_n, and $\sqrt{\Delta}$, respectively.

The identities given in the above theorems may be compared with the ones in [2]. In addition, we can also obtain some interesting equalities. For example, letting $a = 2$, $b = 1$ in (2) and $s = 1$ in (7), respectively, we have

$$\sum_{n=0}^{\infty} \frac{1}{F_{4n+1} - 3/\sqrt{5}} = \sum_{n=0}^{\infty} \frac{1}{L_{2n} - \sqrt{5}} = -1 - \alpha,$$

and letting $s = 2$ in (8), we have

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1} - 3/\sqrt{5}} = -1.$$

3. ESTIMATES OF TWO SERIES

In this section, the summation $\sum_n$ is over all integers n.

Putting $a = b = 1$ and $s = 0$ in the left-hand side of (6), we have

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1} - 2/\sqrt{5}} = \sqrt{5}\sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q^{2n+1}-1)^2},$$

where $q = \alpha^{-1}$. From a classical result (see, e.g., [1] or [6]), we know that

$$\sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q^{2n+1}-1)^2} = \frac{\vartheta_4''}{8\pi^2 \vartheta_4},$$

where

$$\vartheta_4 = \sqrt{-\frac{\pi}{\log q}} \sum_n e^{\pi^2 (n-0.5)^2 (\log q)^{-1}}.$$

Through simple computation, we obtain

$$\sum_{n=0}^{\infty}\frac{1}{F_{2n+1}-2/\sqrt{5}}=\frac{\pi^2\sqrt{5}}{8(\log\alpha)^2}-\frac{\sqrt{5}}{4\log\alpha}-\frac{\pi^2\sum_n(n^2-n)e^{-\pi^2(n-0.5)^2(\log\alpha)^{-1}}}{2(\log\alpha)^2\sum_n e^{-\pi^2(n-0.5)^2(\log\alpha)^{-1}}}.$$

Thus, a good estimate of the series $\sum_{n=0}^{\infty}1/(F_{2n+1}-2/\sqrt{5})$ is given by

$$\frac{\pi^2\sqrt{5}}{8(\log\alpha)^2}-\frac{\sqrt{5}}{4\log\alpha}.$$

Using a similar method, we obtain an estimate of another series. From the following results,

$$\sum_{n=1}^{\infty}\frac{1}{L_{2n}-2}=\sum_{n=1}^{\infty}\frac{q^{2n}}{(q^{2n}-1)^2},$$

where $q=\alpha^{-1}$, and

$$\sum_{n=1}^{\infty}\frac{q^{2n}}{(q^{2n}-1)^2}=\frac{1}{24\pi^2}\left(\frac{\vartheta_2''}{\vartheta_2}+\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_4''}{\vartheta_4}\right)+\frac{1}{24},$$

where (see [1] or [6])

$$\vartheta_2=\sqrt{-\frac{\pi}{\log q}}\sum_n(-1)^n e^{\pi^2n^2(\log q)^{-1}},\quad \vartheta_3=\sqrt{-\frac{\pi}{\log q}}\sum_n e^{\pi^2n^2(\log q)^{-1}},$$

we obtain

$$\sum_{n=1}^{\infty}\frac{1}{L_{2n}-2}\approx\frac{1}{24}-\frac{1}{4\log\alpha}+\frac{\pi^2}{3(\log\alpha)^2}\left(\frac{1}{e^{\pi^2(\log\alpha)^{-1}}+2}-\frac{1}{e^{\pi^2(\log\alpha)^{-1}}-2}+\frac{1}{8}\right)$$
$$\approx\frac{1}{24}-\frac{1}{4\log\alpha}+\frac{\pi^2}{24(\log\alpha)^2}.$$

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REFERENCES

1. G. Almkvist. "A Solution to a Tantalizing Problem." *The Fibonacci Quarterly* **24.4** (1986): 316-22.
2. R. André-Jeannin. "Summation of Certain Reciprocal Series Related to Fibonacci and Lucas Numbers." *The Fibonacci Quarterly* **29.3** (1991):200-04.
3. R. Backstrom. "On Reciprocal Series Related to Fibonacci Numbers with Subscripts in Arithmetic Progression." *The Fibonacci Quarterly* **19.1** (1981):14-21.
4. B. Popov. "On Certain Series of Reciprocals of Fibonacci Numbers." *The Fibonacci Quarterly* **22.3** (1982):261-65.
5. B. Popov. "Summation of Reciprocal Series of Numerical Functions of Second Order." *The Fibonacci Quarterly* **24.1** (1986)17-21.
6. E. T. Whittaker & G. N. Watson. *A Course of Modern Analysis*. Cambridge: Cambridge University Press, 1984.

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