ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-556 ***Proposed by N. Gauthier, Dept. of Physics, Royal Military College of Canada***

Let $f(x)$ and $g(x)$ be continuous and differentiable in the immediate vicinity of $x=a(\neq 0)$ and assume that, for some positive integer k,

$$f^{(n)}(a)=g^{(n)}(a)=0;\quad 0\le n\le k-1.$$

By definition,

$$f^{(n)}(x):=\frac{d^n}{dx^n}f(x)$$

for any continuous and differentiable function $f(x)$. Further, assume that one of the following conditions holds for $n=k$:

a. $f^{(k)}(a)\neq 0,\ g^{(k)}(a)=0;$

b. $f^{(k)}(a)=0,\ g^{(k)}(a)\neq 0;$

c. $f^{(k)}(a)\neq 0,\ g^{(k)}(a)\neq 0;$

Introduce the differential operator $D:=x\frac{d}{dx}$ and define, for m a nonnegative integer,

$$f_m(x):=D^m f(x),\quad g_m(x):=D^m g(x).$$

Prove that

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f_k(a)}{g_k(a)},\quad a\neq 0.$$

H-557 ***Proposed by Stanley Rabinowitz, Westford, MA***

Let $\langle w_n\rangle$ be any sequence satisfying the second-order linear recurrence $w_n=Pw_{n-1}-Qw_{n-2}$, and let $\langle v_n\rangle$ denote the specific sequence satisfying the same recurrence but with the initial conditions $v_0=2$, $v_1=P$.

If k is an integer larger than 1, and $m=\lfloor k/2\rfloor$, prove that, for all integers n,

$$v_n\sum_{i=0}^{m-1}(-Q^n)^i w_{(k-1-2i)n}=w_{kn}-(-Q^n)^m\times\begin{cases}w_0, & \text{if } k \text{ is even,}\\ w_n, & \text{if } k \text{ is odd.}\end{cases}$$

Note: This generalizes problem H-453.

H-558 ***Proposed by Paul S. Bruckman, Berkeley, CA***

Prove the following:

$$\pi = \sum_{n=0}^{\infty}(-1)^n\{6\varepsilon_{10n+1} - 6\varepsilon_{10n+3} - 4\varepsilon_{10n+5} - 6\varepsilon_{10n+7} + 6\varepsilon_{10n+9}\}, \qquad (*)$$

where $\varepsilon_m = \alpha^{-m}/m$.

SOLUTIONS

Count on It!

H-540 ***Proposed by Paul S. Bruckman, Berkeley, CA***
(Vol. 36, no. 2, May 1998)

Consider the sequence $U = \{u(n)\}_{n=1}^{\infty}$, where $u(n) = [n\alpha]$, its characteristic function $\delta_U(n)$, and its counting function $\pi_U(n) \equiv \sum_{k=1}^{n}\delta_U(k)$, representing the number of elements of U that are $\leq n$. Prove the following relationships:

(a) $\delta_U(n) = u(n+1) - u(n) - 1, \ \ n \geq 1;$

(b) $\pi_U(F_n) = F_{n-1}, \ n > 1.$

Solution by H.-J. Seiffert, Berlin, Germany

Let $v(n) = [n\alpha^2]$, $n \in N$, and $V = \{v(n)\}_{n=1}^{\infty}$. It is known (see [1], p. 472) that $U \cap V = \phi$ and $U \cup V = N$. In [1] it is proved that, for all $n \in N$,

$$u(u(n)+1) - u(u(n)) = 2, \qquad (1)$$

$$u(v(n)+1) - u(v(n)) = 1. \qquad (2)$$

The equation

$$u(F_n + 1) = F_{n+1} + 1, \ n \in N, \ n > 1, \qquad (3)$$

is established on page 311 in [2].

Proof of (a): Let $n \in N$, If $n \in U$, then there exists $k \in N$ such that $n = u(k)$. From (1), we get

$$u(n+1) - u(n) - 1 = u(u(k)+1) - u(u(k)) - 1 = 1 = \delta_U(n).$$

If $n \notin U$, then $n \in V$ and $n = v(k)$, where $k \in N$. Hence, by (2),

$$u(n+1) - u(n) - 1 = u(v(k)+1) - u(v(k)) - 1 = 0 = \delta_U(n).$$

Proof of (b): Summing the equations $\delta_U(k) = u(k+1) - u(k) - 1$ over $k = 1, \ldots, n$ and using $u(1) = 1$ gives

$$\pi_U(n) = u(n+1) - n - 1, \ n \in N. \qquad (4)$$

If $n > 1$, then by (3) and (4), $\pi_U(F_n) = u(F_n + 1) - F_n - 1 = F_{n+1} - F_n = F_{n-1}$.

References:

1. V. E. Hoggatt, Jr. & A. P. Hillman. "A Property of Wythoff Pairs." *The Fibonacci Quarterly* **16.5** (1978):472.
2. V. E. Hoggatt, Jr., & M. Bicknell-Johnson. "Representations of Integers in Terms of Greatest Integer Functions and the Golden Section Ratio." *The Fibonacci Quarterly* **17.4** (1979):306-18.

Also solved by the proposer.

Just Continue

H-541 ***Proposed by Stanley Rabinowitz, Westford, MA***
(Vol. 36, no. 2, May 1998)

The simple continued fraction expansion for F_{13}^5 / F_{12}^5 is

$$11+\cfrac{1}{11+\cfrac{1}{375131+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{9+\cfrac{1}{11}}}}}}}}}}}}}}$$

which can be written more compactly using the notation $[11,11,375131,1,1,1,1,1,1,1,1,1,2,9,11]$. To be even more concise, we can write this as $11^2,375131,1^9,2,9,11]$, where the superscript denotes the number of consecutive occurrences of the associated number in the list.

If $n>0$, prove that the simple continued fraction expansion for $(F_{10n+3}/F_{10n+2})^5$ is $11^{2n},x,$ ${}^{10n-1},2,9,11^{2n-1}]$, where x is an integer, and find x.

Solution by Paul S. Bruckman, Berkeley, CA

We begin with the well-known isomorphism between simple continued fractions and 2×2 matrices, namely:

$$\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_3 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_m & 1 \\ 1 & 0 \end{pmatrix}=\begin{pmatrix} p_m & p_{m-1} \\ q_m & q_{m-1} \end{pmatrix},$$

where p_m/q_m is the m^{th} convergent of the simple continued fraction denoted as $[a_1,a_2,a_3,\ldots,a_m]$. Normally, we will restrict the a_i's to be positive integers. As a particular case, if all the a_i's are equal (say to a), this result simplifies to

$$\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}^m=\begin{pmatrix} \Phi_{m+1} & \Phi_m \\ \Phi_m & \Phi_{m-1} \end{pmatrix}, \text{ where } \Phi_m=\Phi_m(a)=\frac{r^m-s^m}{r-s},$$

$$r=r(a)=\tfrac{1}{2}\{a+\theta\},\ \ s=s(a)=\tfrac{1}{2}\{a-\theta\}, \text{ and } \theta=\theta(a)=(a^2+4)^{1/2}.$$

It is also true that $\Phi_{m+2}=a\Phi_{m+1}+\Phi_m$, $m=0,1,2,\ldots$, with $\Phi_0=0$, $\Phi_1=1$. We note that $\Phi_m(1)=F_m$, and also that $r(11)=\tfrac{1}{2}(11+5\sqrt{5})=\alpha^5$, $s(11)=\tfrac{1}{2}(11-5\sqrt{5})=\beta^5$; therefore,

$$\Phi_m(11)=1/5F_{5m}. \tag{1}$$

We will also utilize the following common identities:

$$5F_uF_v=L_{u+v}-(-1)^u L_{v-u};\ \ F_uL_v=F_{u+v}-(-1)^u F_{v-u};\ \ L_uL_v=L_{u+v}+(-1)^u L_{v-u};$$

and

$$25(F_m)^5 = F_{5m} - 5(-1)^m F_{3m} + 10F_m. \tag{2}$$

For brevity, let $\rho_n = (F_{10n+3} / F_{10n+2})^5$, $n = 1, 2, \ldots$. Also, we *assume* that $\rho_n = [11^{2n}, \xi]$, where $\xi = \xi_n$ is not necessarily an integer. This implies that $x = x_n = \lfloor \xi \rfloor$ (here "$\lfloor\ \rfloor$" denotes the "greatest integer" function). Using the formula in (2), we find that

$$\rho_n = (F_{50n+15} + 5F_{30n+9} + 10F_{10n+3}) / (F_{50n+10} - 5F_{30n+6} + 10F_{10n+2}). \tag{3}$$

Also

$$\begin{pmatrix} 11 & 1 \\ 1 & 0 \end{pmatrix}^{2n} = 1/5\begin{pmatrix} F_{10n+5} & F_{10n} \\ F_{10n} & F_{10n-5} \end{pmatrix}, \text{ using (1).}$$

Thus,

$$\begin{pmatrix} 11 & 1 \\ 1 & 0 \end{pmatrix}^{2n}\begin{pmatrix} \xi & 1 \\ 1 & 0 \end{pmatrix} = 1/5\begin{pmatrix} \xi F_{10n+5} + F_{10n} & F_{10n+5} \\ \xi F_{10n} + F_{10n-5} & F_{10n} \end{pmatrix}.$$

Then we require that $\rho_n = (\xi F_{10n+5} + F_{10n}) / (\xi F_{10n} + F_{10n-5})$. Now we substitute the formula in (3), cross-multiply, and simplify, using the multiplication identities previously indicated. After a tedious but straightforward computation, we obtain the following result:

$$\xi_n = 5F_{20n+5} + 6 + (F_{20n+4} + 2) / F_{20n+5}. \tag{4}$$

Note that, if $n > 0$, the fractional part of ξ_n lies in the interval $(0, 1)$, as we would expect. Thus, our earlier assumption is justified, and we conclude that

$$x_n = 5F_{20n+5} + 6. \tag{5}$$

By comparison with the desired expression, it remains to verify that

$$F_{20n+5} / (F_{20n+4} + 2) = [1^{10n-1}, 2, 9, 11^{2n-1}]. \tag{6}$$

In turn, it suffices to show the following:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{10n-1}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 9 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 11 & 1 \\ 1 & 0 \end{pmatrix}^{2n-1} = C\begin{pmatrix} F_{20n+5} & * \\ F_{20n+4} + 2 & * \end{pmatrix}, \tag{7}$$

where C is some constant independent of n and the "*" matrix entries are not important to know. Based on our previous results, the left member of (7) is expanded as follows,

$$1/5\begin{pmatrix} F_{10n} & F_{10n-1} \\ F_{10n-1} & F_{10n-2} \end{pmatrix}\begin{pmatrix} 19 & 2 \\ 9 & 1 \end{pmatrix}\begin{pmatrix} F_{10n} & F_{10n-5} \\ F_{10n-5} & F_{10n-10} \end{pmatrix}$$

$$= 1/5\begin{pmatrix} 19F_{10n} + 9F_{10n-1} & F_{10n+2} \\ 19F_{10n-1} + 9F_{10n-2} & F_{10n+1} \end{pmatrix}\begin{pmatrix} F_{10n} & F_{10n-5} \\ F_{10n-5} & F_{10n-10} \end{pmatrix} = 1/25\begin{pmatrix} A_n & * \\ B_n & * \end{pmatrix},$$

where (after simplification):

$$A_n = 19L_{20n} + 9L_{20n-1} + L_{20n-3} = 5F_{20n+5},$$

$$B_n = 19L_{20n-1} + 9L_{20n-2} + L_{20n-4} + 19 - 27 + 18 = 5F_{20n+4} + 10.$$

Thus, (7) is verified (with $C = 1/5$) and the proof is complete. □

Note: This result invites generalizations. If $r_n = F_{n+1} / F_n$, we would like to find similar results involving $(r_n)^k$ for various values of n and k. The following result (using the proposer's notation) is well known:

$$r_n = [1^n]. \tag{8}$$

The following may also be shown:

$$(r_n)^2 = [2, 1^{n-3}, 3, 1^{n-2}], \quad \text{if } n \geq 3. \tag{9}$$

Also, we may derive the following relations, valid for $n \geq 1$:

$$\begin{aligned}(r_{3n+1})^3 &= [4^n, 1, 1, 1, 4^{n-1}, 2, 2, 1^{3n-2}];\\(r_{3n+2})^3 &= [4^n, 10, 4^{n-1}, 2, 2, 1^{3n-1}];\\(r_{3n+3})^3 &= [4^n, 3, 2, 3, 4^{n-1}, 2, 2, 1^{3n}].\end{aligned} \tag{10}$$

These kinds of expressions become more complicated for increasing values of k, and apparently require separate treatment for the different values of n (mod k). The matrix method indicated above seems to be the most efficient way to handle such problems. □

Sum Problem!

H-542 *Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 36, no. 4, August 1998)*

Define the sequence $(c_k)_{k\geq 1}$ by

$$c_k = \begin{cases} 1 & \text{if } k \equiv 2 \pmod 5, \\ -1 & \text{if } k \equiv 3 \pmod 5, \\ 0 & \text{otherwise.} \end{cases}$$

Show that, for all positive integers n:

$$\frac{1}{n}\sum_{k=1}^{n} k\binom{2n}{n-k}c_k = F_{2n-2}; \tag{1}$$

$$\frac{1}{2n-1}\sum_{k=1}^{2n-1}(-1)^k k\binom{4n-2}{2n-k-1}c_k = 5^{n-1}F_{2n-2}; \tag{2}$$

$$\frac{1}{2n}\sum_{k=1}^{2n}(-1)^k k\binom{4n}{2n-k}c_k = 5^{n-1}L_{2n-1}. \tag{3}$$

Solution by the proposer

We consider the Fibonacci polynomials defined by $F_0(x) = 0$, $F_1(x) = 1$, $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$, for $n \geq 0$. It is known that

$$\sum_{k=1}^{\infty} F_k(x)z^k = \frac{z}{1-xz-z^2}, \text{ for small } |z|.$$

Replacing z by iz, $i = \sqrt{(-1)}$, and taking $x = i\alpha$, resp. $x = i\beta$, gives

$$\sum_{k=1}^{\infty} F_k(i\alpha)(iz)^k = \frac{iz}{1+\alpha z+z^2}, \text{ resp. } \sum_{k=1}^{\infty} F_k(i\beta)(iz)^k = \frac{iz}{1+\beta z+z^2}.$$

Subtracting the first from the second equation and dividing the resulting equation by $i\sqrt{5}$ yields

$$\sum_{k=1}^{\infty}\frac{i^{k-1}}{\sqrt{5}}(F_k(i\beta) - F_k(i\alpha))z^k = \frac{z^2}{1+z+z^2+z^3+z^4}.$$

The sequence $(c_k)_{k\geq 1}$ has the generating function

$$\sum_{k=1}^{\infty} c_k z^k = \sum_{j=0}^{\infty} (z^{5j+2} - z^{5j+3}) = \frac{z^2(1-z)}{1-z^5}, \; |z| < 1.$$

Since $1+z+z^2+z^3+z^4 = (1-z^5)/(1-z)$, comparing coefficients gives

$$c_k = \frac{i^{k-1}}{\sqrt{5}}(F_k(i\beta) - F_k(i\alpha)), \; k \in N. \tag{4}$$

From H-518, we know that, for all complex numbers x and y and all positive integers n,

$$\sum_{k=1}^{n} \binom{2n}{n-k} F_k(x) F_k(y) = (x-y)^{n-1} F_n\left(\frac{xy+4}{x-y}\right).$$

Taking $y = 2i$ and using $F_k(2i) = ki^{k-1}$, we find

$$\sum_{k=1}^{n} k \binom{2n}{n-k} i^{k-1} F_k(x) = n(2+ix)^{n-1}. \tag{5}$$

From (4) and (5), we obtain

$$\sum_{k=1}^{n} k \binom{2n}{n-k} c_k = n \frac{(2-\beta)^{n-1} - (2-\alpha)^{n-1}}{\sqrt{5}}.$$

Using $2-\beta = \alpha^2$ and $2-\alpha = \beta^2$ and the Binet form of F_{2n-2} gives the first desired identity (1).

Since $F_k(-x) = (-1)^{k-1} F_k(x)$, we also find

$$\begin{aligned}
&\sum_{k=1}^{n} (-1)^k k \binom{2n}{n-k} c_k \\
&= \sum_{k=1}^{n} k \binom{2n}{n-k} \frac{i^{k-1}}{\sqrt{5}} (F_k(-i\alpha) - F_k(-i\beta)) \\
&= n \frac{(2+\alpha)^{n-1} - (2+\beta)^{n-1}}{\sqrt{5}}.
\end{aligned}$$

Since $2+\alpha = \sqrt{5}\alpha$ and $2+\beta = -\sqrt{5}\beta$, if we replace n by $2n-1$ and n by $2n$, we easily obtain (2) and (3), respectively.

Also solved by P. Bruckman

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