AN EXTENSION OF AN OLD PROBLEM OF DIOPHANTUS AND EULER

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Diophantus studied the following problem: Find three (rational) numbers such that the product of any two increased by the sum of those two gives a square. He obtained the solutions $\{4, 9, 28\}$ and $\{\frac{3}{10}, \frac{21}{5}, \frac{7}{10}\}$ (see [3]). Euler treated the same problem with four numbers (see [2]). He found the solution $\{\frac{65}{224}, \frac{9}{224}, \frac{9}{56}, \frac{5}{2}\}$. Indeed, we have

$$\frac{65}{224}\cdot\frac{9}{224}+\frac{65}{224}+\frac{9}{224}=\left(\frac{131}{224}\right)^2, \quad \frac{65}{224}\cdot\frac{9}{56}+\frac{65}{224}+\frac{9}{56}=\left(\frac{79}{112}\right)^2,$$

$$\frac{65}{224}\cdot\frac{5}{2}+\frac{65}{224}+\frac{5}{2}=\left(\frac{15}{8}\right)^2, \quad \frac{9}{224}\cdot\frac{9}{56}+\frac{9}{224}+\frac{9}{56}=\left(\frac{51}{112}\right)^2,$$

$$\frac{9}{224}\cdot\frac{5}{2}+\frac{9}{224}+\frac{5}{2}=\left(\frac{13}{8}\right)^2, \quad \frac{9}{56}\cdot\frac{5}{2}+\frac{9}{56}+\frac{5}{2}=\left(\frac{7}{4}\right)^2.$$

In the present paper we will construct the set of **five** numbers with the above property.

Let $\{x_1, \ldots, x_m\}$ be the set of rational numbers such that $x_ix_j+x_i+x_j$ is a perfect square for all $1\le i<j\le m$. Since

$$x_ix_j+x_i+x_j=(x_i+1)(x_j+1)-1,$$

if we put $x_i+1=a_i$, $i=1,\ldots,m$, we obtain the set $\{a_i, \ldots, a_m\}$ with the property that the product of its any two distinct elements diminished by 1 is a perfect square. Such a set is called *a (rational) Diophantine m-tuple with the property* $D(-1)$ (see [4], p. 75). If a_i's are positive integers, such a set is also called *a* P_{-1}*-set of size m*. The conjecture is that there does not exist a P_{-1}-set of size 4. Let us mention that in [1], [6], and [7] it was proved that some particular P_{-1}-sets of size 3 cannot be extended to a P_{-1}-set of size 4. In [5], some consequences of the above conjecture were considered.

We will derive a two-parametric formula for Diophantine quintuples and, as a consequence, we will obtain a rational Diophantine quintuple with the property $D(-1)$.

We will consider quintuples of the form $\{A, B, C, D, x^2\}$ with the property $D(\alpha x^2)$, where A, B, C, D, x, and α are integers. Furthermore, we will use the following simple result known already to Euler: If $BC+n=k^2$, then the set $\{B, C, B+C\pm 2k\}$ has the property $D(n)$.

Therefore, if we assume that

$$BC+\alpha x^2=k^2, \quad A=B+C-2k, \quad D=B+C+2k,$$

then the set $\{A, B, C, D, x^2\}$ has the property $D(\alpha x^2)$ if and only if $AD+\alpha x^2$ is a perfect square. Hence, we reduced the original $\binom{5}{2}=10$ conditions to only two conditions:

$$(b^2-\alpha)(c^2-\alpha)+\alpha x^2=k^2, \tag{1}$$

$$(a^2-\alpha)(d^2-\alpha)+\alpha x^2=y^2. \tag{2}$$

Our assumptions

$$(b^2-\alpha)+(c^2-\alpha)-2k=a^2-\alpha, \quad (b^2-\alpha)+(c^2-\alpha)+2k=d^2-\alpha$$

imply that $4k=(d+a)(d-a)$. Let $d+a=2p$ and $d-a=2r$. This implies that $k=pr$ and

$$b^2+c^2-\alpha=\frac{1}{2}(a^2+d^2)=p^2+r^2. \tag{3}$$

Let us rewrite condition (2) in the form $(ad-\alpha)^2-\alpha(d-a)^2=y^2-\alpha x^2$. Thus, we may take

$$y=ad-\alpha, \quad x=d-a=2r. \tag{4}$$

Substituting (3) and (4) into (1), we obtain

$$p^2r^2-b^2c^2=4\alpha r^2-\alpha(b^2+c^2-\alpha)=\alpha(3r^2-p^2). \tag{5}$$

At this point we make the further assumption [motivated by (3) and (5)]:

$$b+c=p+r. \tag{6}$$

Now (3) implies

$$pr-bc=\frac{\alpha}{2}, \tag{7}$$

and (5) implies

$$pr+bc=2(3r^2-p^2). \tag{8}$$

Adding (7) and (8) yields

$$\alpha=4p^2+4pr-12r^2. \tag{9}$$

From (6) and (7), we conclude that b and c are the solutions of the quadratic equation

$$z^2-(p+r)z+\left(pr-\frac{\alpha}{2}\right)=0.$$

The discriminant of this equation has to be a perfect square. Thus,

$$(p-r)^2+2\alpha=q^2. \tag{10}$$

Substituting (9) into (10) we have, finally,

$$(3p+r)^2-24r^2=q^2. \tag{11}$$

Hence, we reduce our problem to the solving of (11). However, the general solution of the equation $u^2-24v^2=w^2$ with $(u,v,w)=1$ is given by

$$u=e^2+6f^2, \quad v=ef, \quad w=|e^2-6f^2|$$

or

$$u=2e^2+3f^2, \quad v=ef, \quad w=|2e^2-3f^2|$$

(see [8], p. 225). Thus, we have proved

Theorem 1: If $e\equiv 0 \pmod 3$ or $e\equiv f \pmod 3$, then the set

$$\begin{aligned}&\{\tfrac{1}{3}(e^2+6ef-18f^2)(2f^2+2ef-e^2), \tfrac{1}{3}e^2(e+5f)(3f-e),\\ &f^2(e-2f)(5e+6f), \tfrac{1}{3}(e^2+4ef-6f^2)(6f^2+4ef-e^2), 4e^2f^2\}\end{aligned} \tag{12}$$

has the property $D\left(\frac{16}{9}e^2f^2(e^2-ef-3f^2)(e^2+2ef-12f^2)\right)$, and the set

$$\begin{aligned}&\{\tfrac{1}{3}(9f^2+6ef-2e^2)(2e^2+2ef-f^2), \tfrac{1}{3}e^2(5f-2e)(2e+3f),\\ &f^2(e+f)(5e-3f), \tfrac{1}{3}(3f^2+4ef-2e^2)(2e^2+4ef-3f^2), 4e^2f^2\}\end{aligned} \tag{13}$$

has the property $D\left(\frac{16}{9}e^2f^2(e^2-ef-3f^2)(4e^2+2ef-3f^2)\right)$.

Substituting $e=5$ and $f=2$ in (12), we obtain the following two corollaries.

Corollary 1: The set $\{\frac{13}{40}, \frac{25}{8}, \frac{37}{10}, 10, \frac{533}{40}\}$ is a rational Diophantine quintuple with the property $D(-1)$.

Corollary 2: The five numbers $-\frac{27}{40}, \frac{17}{8}, \frac{27}{10}, 9, \frac{493}{40}$ have the property that the product of any two of them increased by the sum of those two gives a perfect square.

REFERENCES

1. E. Brown. "Sets in Which $xy+k$ Is Always a Square." *Math. Comp.* **45** (1985):613-20.
2. L. E. Dickson. *History of the Theory of Numbers* **2:**518-19. New York: Chelsea, 1966.
3. Diophantus of Alexandria. *Arithmetics and the Book of Polygonal Numbers*, pp. 85-86, 215-17. Ed. I. G. Bashmakova. Moscow: Nauka, 1974. (In Russian.)
4. A. Dujella. "On Diophantine Quintuples." *Acta Arith.* **81** (1997):69-79.
5. A. Dujella. "On the Exceptional Set in the Problem of Diophantus and Davenport." In *Applications of Fibonacci Numbers* 7. Dordrecht: Kluwer, 1998.
6. K. S. Kedlaya. "Solving Constrained Pell Equations." *Math. Comp* **67** (1998):833-42.
7. S. P. Mohanty & A. M. S. Ramasamy. "The Simultaneous Diophantine Equations $5y^2-20=x^2$ and $2y^2+1=z^2$." *J. Number Theory* **18** (1984):356-59.
8. T. Nagell. *Introduction to Number Theory*. Stockholm: Almqvist; New York: Wiley, 1951.

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