ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-559 ***Proposed by N. Gauthier, Royal Military College of Canada***

Let n and q be nonnegative integers and show that:

a. $$S_n(q) := \sum_{k=1}^{n} \frac{1}{2\cos(2\pi k/n) + (-1)^{q+1}L_{2q}} = \frac{(-1)^{q+1} n L_{qn}}{5F_{2q}F_{qn}}.$$

b. $$s_n(q) := \sum_{k=1}^{n} \frac{1}{0.8\sin^2(2\pi k/n) + F_{2q}^2}$$
$$= \frac{nL_{2qn}}{F_{2q}L_{2q}F_{qn}L_{qn}}, \quad n \text{ odd},$$
$$= \frac{nL_{qn}}{F_{2q}L_{2q}F_{qn}}, \quad n \text{ even}.$$

L_n and F_n are Lucas and Fibonacci numbers.

H-560 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Define the sequences of Fibonacci and Lucas polynomials by

$$F_0(x) = 0,\ F_1(x) = 1, \text{ and } F_{n+1}(x) = xF_n(x) + F_{n-1}(x),\ n \in N,$$

and

$$L_0(x) = 2,\ L_1(x) = x, \text{ and } L_{n+1}(x) = xL_n(x) + L_{n-1}(x),\ n \in N,$$

respectively. Show that, for all complex numbers x and all positive integers n,

$$\sum_{k=0}^{[n/2]} \frac{n}{n-k}\binom{n-k}{k} x^k F_{3k}(x) = F_{2n}(x) + (-x)^n F_n(x)$$

and

$$\sum_{k=0}^{[n/2]} \frac{n}{n-k}\binom{n-k}{k} x^k L_{3k}(x) = L_{2n}(x) + (-x)^n L_n(x).$$

SOLUTIONS

Continuing...

H-543 ***Proposed by David M. Bloom, Brooklyn College of CUNY, Brooklyn, NY (Vol. 36, no. 4, August 1998)***

Find all positive nonsquare integers d such that, in the continued fraction expansion

$$\sqrt{d} = \left[n; \overline{a_1, \ldots, a_{r-1}, 2n}\right],$$

we have $a_1 = \cdots = a_{r-1} = 1$. (This includes the case $r = 1$ in which there are no a's.)

Solution by Charles K. Cook, University of South Carolina Sumter, Sumter, SC

For the case $[n; \overline{2n}]$, it is known (see [1], p. 80) that $x = [\overline{2n}]$ satisfies $x^2 = 2nx + 1$. Thus, $x = n + \sqrt{n^2+1}$ and so

$$\sqrt{d} = n + \frac{1}{n + \sqrt{n^2+1}}$$

which simplifies to $d = n^2 + 1$.

Setting y equal to the periodic expansion and recovering a relationship for y using the usual formal manipulations on the continued fraction representation

$$y = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots + \cfrac{1}{2n + \cfrac{1}{y}}}}$$

yields the following equations for y:

$$\begin{array}{ll}
y = \left[0; \overline{1, 2n}\right] & 2ny^2 - 2ny - 1 = 0 \\
y = \left[0; \overline{1, 1, 2n}\right] & (2n+1)y^2 - 4ny - 2 = 0 \\
y = \left[0; \overline{1, 1, 1, 2n}\right] & (4n+1)y^2 - 6n - 3 = 0 \\
y = \left[0; \overline{1, 1, 1, 1, 2n}\right] & (6n+2)y^2 - 10ny - 5 = 0 \\
y = \left[0; \overline{1, 1, 1, 1, 1, 2n}\right] & (10n+3)y^2 - 16ny - 8 = 0
\end{array}$$

and, in general, if F_m is the m^{th} Fibonacci number, then $y = [0; \overline{m-\text{ones},\ 2n}]$ and y satisfies $(2nF_m + F_{m-1})y^2 - 2nF_{m+1}y - F_{m+1} = 0$, which can be shown by a routine inductive argument.

Thus,

$$n^2 + \frac{(2n-1)F_m + F_{m+1}}{F_{m+1}}$$

must be integral. So both

$$n^2 + 1 + \frac{(2n-1)F_m}{F_{m+1}} \quad \text{and} \quad \frac{(2n-1)F_m}{F_{m+1}}$$

are integral.

However, $\gcd(F_m, F_{m+1}) = 1$, so $2n \equiv 1 \pmod{F_{m+1}}$. Hence, F_{m+1} must be odd. Therefore, $\gcd(2, F_{m+1}) = 1$, and the linear congruence $2n \equiv 1 \pmod{F_{m+1}}$ always has a solution. Thus, if m is the number of ones in the continued fraction expansion, it follows that

$$d = n^2 + 1 + \frac{(2n-1)F_m}{F_{m+1}}$$

provided F_{m+1} is odd.

A few solutions are shown in the table below.

m	F_m	$n = n(k), k \geq 1$	n values	$d = d(k)$	d values
0	1	k	1, 2, 3, ...	$k^2 + 1$	2, 5, 10, ...
1	1	k	1, 2, 3, ...	$k^2 + 2k$	3, 8, 15, ...
2	2	None	None	$k^2 + k + \frac{1}{2}$	None
3	3	$3k - 1$	2, 5, 8, ...	$9k^2 - 2k$	7, 32, 75, ...
4	5	$5k - 2$	3, 8, 13, ...	$25k^2 - 14k + 2$	13, 74, 185, ...
5	8	None	None	$k^2 + (10k + 3)/8$	None
6	13	$13k - 6$	7, 20, 33, ...	$169k^2 - 140k + 29$	58, 425, 1130, ...
7	21	$21k - 10$	11, 32, 53, ...	$441k^2 - 394k + 88$	135, 1064, 2875, ...
8	34	None	None	$k^2 + (42k + 13)/34$	None

Reference

1. C. D. Olds. *Continued Fractions.* Washington, D.C.: The Mathematical Association of America, 1963.

Also solved by P. Bruckman, A. Tuyl, and the proposer.

Primes and FPP's

H-544 ***Proposed by Paul S. Bruckman, Berkeley, CA***
(Vol. 36, no. 4, August 1998)

Given a prime $p > 5$ such that $Z(p) = p + 1$, suppose that $q = \frac{1}{2}(p^2 - 3)$ and $r = p^2 - p - 1$ are primes with $Z(q) = q + 1$, $Z(r) = \frac{1}{2}(r - 1)$. Prove that $n = pqr$ is a FPP (see previous proposals for definitions of the Z-function and of FPP's).

Solution by the proposer

For all natural m such that $\gcd(m, 10) = 1$, let ε_m denote the Jacobi symbol $(5/m)$, and $m' = m - \varepsilon_m$. If s is any prime $\neq 2, 5$, it is well known that $Z(s) \mid s'$. We then see that $\varepsilon_p = \varepsilon_q = -1$, $\varepsilon_r = \varepsilon_n = \varepsilon_p \varepsilon_q \varepsilon_r = +1$. Thus, $p \equiv \pm 3$, $q \equiv \pm 3$, $r \equiv \pm 1$, $n \equiv \pm 1 \pmod{10}$.

Now, if s is any prime $\neq 2, 5$ and $a(s) = s' / Z(s)$, then $a(s)$ and $\frac{1}{2}(s-1)$ have the same parity (see this journal, Problem H-494, Vol. 33, no. 1, Feb. 1995; solution in Vol. 34, no. 2, Aug. 1996, pp. 190-91). Since $a(p) = a(q) = 1$, $a(r) = 2$, it follows that $p \equiv q \equiv 3$, $r \equiv n \equiv 1 \pmod 4$. Also, $3^2 - 3 - 1 = 5$, which shows that r cannot be prime if $p \equiv 3 \pmod{20}$. Therefore, $p \equiv 7 \pmod{20}$; this in turn implies that $q \equiv 3$, $r \equiv n \equiv 1 \pmod{20}$.

Next, we see that $Z(q) = \frac{1}{2}(p^2 - 1)$, $Z(r) = \frac{1}{2}(p+1)(p-2)$. Then

$$Z(n) = \operatorname{lcm}\{Z(p), Z(q), Z(r)\} = \tfrac{1}{2}(p^2 - 1)(p - 2).$$

In order to show that n is a FPP, it suffices to show that $n - 1 = n' \equiv 0 \pmod{Z(n)}$. Now

$$pq-r=\tfrac{1}{2}(p^3-2p^2-p+2)=\tfrac{1}{2}(p^2-1)(p-2)=Z(n);$$

hence, $pq\equiv r \pmod{Z(n)}$. Then $n\equiv r^2 \pmod{Z(n)}$. Next, $r+1=p(p-1)$, $r-1=(p+1)(p-2)$, whence $r^2-1=p(p^2-1)(p-2)=2pZ(n)\equiv 0 \pmod{Z(n)}$. Thus, $n'=n-1\equiv r^2-1\equiv 0$ (mod $Z(n)$), which shows that n is a FPP. Q.E.D.

Note: The smallest FPP satisfying the above conditions is $7\cdot 23\cdot 41$ $(p=7)$.

Also solved by H.-J. Seiffert.

An Interesting Equation

H-553 ***Proposed by Paul S. Bruckman, Berkeley, CA***
(Vol. 37, no. 3, August 1999)

The following Diophantine equation has the trivial solution $(A, B, C, D)=(A, A, A, 0)$:

$$A^3+B^3+C^3-3ABC=D^k, \text{ where } k \text{ is a positive integer.} \tag{1}$$

Find nontrivial solutions of (1), i.e., with all quantities positive integers.

Solution (1) by the proposer

Let

$$\theta=\exp\left(\tfrac{2}{3}i\pi\right), \tag{2}$$

$$K(a,b,c)=a^3+b^3+c^3-3abc. \tag{3}$$

As we may easily verify:

$$K(a,b,c)=s(a,b,c)\cdot s(a,b\theta,c\theta^2)\cdot s(a,b\theta^2,c\theta), \tag{4}$$

where

$$s(a,b,c)=a+b+c. \tag{5}$$

Given U, V, W positive integers, where at least two of them are distinct, let

$$X=(s(U,V,W))^k,\ Y=(s(U,V\theta,W\theta^2))^k,\ Z=(s(U,V\theta^2,W\theta))^k. \tag{6}$$

From (4), it follows that

$$XYZ=(K(U,V,W))^k. \tag{7}$$

Now define the following quantities:

$$A=\tfrac{1}{3}s(X,Y,Z),\ B=\tfrac{1}{3}s(X,Y\theta^2,Z\theta),\ C=\tfrac{1}{3}s(X,Y\theta,Z\theta^2). \tag{8}$$

Again using (4), we see that

$$27ABC=K(X,Y,Z). \tag{9}$$

We now employ the following well-known expression:

$$\tfrac{1}{3}(1+\theta^r+\theta^{2r})=\begin{cases}1 & \text{if } 3\mid r,\\ 0 & \text{if } 3\nmid r.\end{cases} \tag{10}$$

By trinomial expansion of the quantities defined in (8), implementing (6) and (10), we obtain the following expressions:

$$A=F_0(U,V,W),\ B=F_1(U,V,W),\ C=F_2(U,V,W), \tag{11}$$

where

$$F_j(U,V,W) = \sum_{\substack{f+g+h=k \\ g-h\equiv j \pmod 3}} \binom{k}{f,g,h} U^f V^g W^h, \; j=0,1,2, \tag{12}$$

$$\text{and } \binom{k}{f,g,h} \text{ is a trinomial coefficient } = \frac{k!}{f!g!h!}.$$

From (12), it is clear that A, B, and C are positive integers. We may also easily verify the following inverse relations:

$$X = s(A,B,C), \; Y = s(A, B\theta, C\theta^2), \; Z = s(A, B\theta^2, C\theta). \tag{13}$$

Again using (4), this implies

$$XYZ = K(A,B,C). \tag{14}$$

From (7) and (14), it follows that

$$K(A,B,C) = (K(U,V,W))^k. \tag{15}$$

Thus, by reference to (1), we see that we may set

$$D = K(U,V,W). \tag{16}$$

Accordingly, solutions (A,B,C,D) of (1) are given by (11) and (16); alternatively, A, B, and C may be obtained indirectly from (8) and (6).

Note that the restriction that U, V, and W be not all identical ensures that Y and Z are positive, as of course is X. Then, from (7) and (16), it follows that $D > 0$, which avoids trivial solutions.

Solution (2) by John Jaroma and Rajib Rahman, Gettysburg College, Gettysburg, PA

After a brief historical background, we will show that, in fact, there are an infinite number of solutions of (1), subject to (2):

$$A^3 + B^3 + C^3 - 3ABC = D^k; \tag{1}$$

$$A, B, C, D \in \{1, 2, \ldots\} \quad \text{and} \quad k \in \{2, 3, \ldots\}. \tag{2}$$

First, in terms of a historical perspective, it appears that Diophantine equations involving cubic terms have generated considerable interest. For example, in 1847, J. J. Sylvester provided sufficient conditions for the insolubility in integers of the equation

$$Ax^3 + By^3 + Cz^3 = Dxyz. \tag{3}$$

Moreover, Sylvester was able to prove that whenever (3) is insoluble, there must exist an entire family of related equations equally insoluble. His motivation for studying such equations was to break ground in the area of third-degree equations. Ultimately, Sylvester had hoped to open a new field in connection with Fermat's Last Theorem.

Today, cubic equations continue to command a great deal of attention. For instance, although we know that every number (with the possible exception of those in the form $9n \pm 4$) can be expressed as the sum of four cubes, it is still not known whether every number can be expressed as the sum of four cubes with two of the cubes equal. Stated algebraically, we would like to know, if given any k, do integral solutions exist for the Diophantine equation

$$A^3 + B^3 + 2C^3 = k. \tag{4}$$

($k = 76$ is the first of many values of k for which an integral solution is not known.)

Perhaps an even more difficult problem exists in the question whether numbers not of the form $9n \pm 4$ can be expressed as the sum of three cubes; that is, does the equation

$$A^3 + B^3 + C^3 = k \tag{5}$$

have a solution in integers $\forall k \neq 9n \pm 4$? The first known value of k for which the problem becomes open is $k = 30$. Furthermore, even if we restrict ourselves to the specific case $k = 3$, we do not know whether (1, 1, 1) and (4, 4, −5) are the only two solutions of (5).

It is likely that Diophantine equations will continue to be an area of research for some time to come, for we know that, given an arbitrary Diophantine equation, there cannot exist an algorithm which in a finite number of steps will decide its solvability. Hilbert's Tenth Problem was demonstrated to be unsolvable by Yuri Matiyasevich in 1970.

Consider the following infinite sets:

(I) $p \in \{1, 2, \dots\}$, $k = 3p + 1$, $n_1, n_2 \in \{1, 2, \dots\} : n_1 / n_2 \in \{2, 3, \dots\}$,
$D = 1 + n_1^3 + (n_1 / n_2)^3 - 3(n_1 / n_2)n_1$,
$A = D^p$, $B = (n_1 / n_2)A$, $C = n_1 A = (n_2 B)$.

(II) $k = 2$, $n \in \{1, 2, \dots\}$,
$B = D = 9n^2$, $A = D - n$, $C = D + n$.

Remark: We have ignored the case where $p = 0$, for this would imply that $k = 1$ and it would then be trivial to produce infinitely many solutions of (1).

Proposition: Sets (I) and (II) represent disjoint families of solutions of (1) satisfying (2).

Proof: We first prove that (I) and (II) are disjoint families of solutions of (1). Since elements of (I) and (II) are ordered 4-tuples of the form (A, B, C, D) and $p \in \{1, 2, \dots\}$, it follows immediately that (I) and (II) are disjoint as $3p + 1 \neq 2$.

Now, to show that (I) represents an infinite set of solutions of (1), we let $n = n_2$. Hence, $n_1 = bn$ for some $b \in \{2, 3, \dots\}$ and

$$D = 1 + b^3 + b^3 n^3 - 3b^2 n, \quad B = bA, \quad C = nbA = nB. \tag{6}$$

Substituting (6) into (1), we get

$$D^{3p} + b^3 D^{3p} + n^3 b^3 D^{3p} - 3nb^2 D^{3p} = D^{3p+1}. \tag{7}$$

Rewriting (7), we obtain

$$D^{3p}(1 + b^3 + n^3 b^3 - 3nb^2) = D^{3p+1}. \tag{8}$$

Thus, (8) is true if and only if $1 + b^3 + b^3 n^3 - 3b^2 n = D$. By (6), the result follows immediately.

Finally to show that (II) is also an infinite family of solutions of (1), we infer from (II) that $B = D = n + A$ and $C = 2n + A$.

Substituting these quantities and the hypothesis that $k = 2$ into (1), we obtain

$$A^3 + (n + A)^3 + (2n + A)^3 - 3A(n + A)(2n + A) = (n + A)^2. \tag{9}$$

Simplifying (9), we obtain $9n^2(n + A) = (n + A)^2$. It now follows that (II) is a set of solutions of (1) if and only if $9n^2 - n - A = 0$. But, by hypothesis, $A = D - n = 9n^2 - n$, and this produces the desired result.

Also solved by B. Beasley, C. Cook, and H.-J. Seiffert.

❖❖❖