ON PRIMES IN THE FIBONACCI SEQUENCE

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(*Submitted May 1998*)

It is well known that primes can occur in the Fibonacci sequence only for prime indices, the only exception being $F_4 = 3$. This follows from the fact that for any two positive integers k and n, F_n divides F_{kn}. I could not locate the earliest reference to that result, but page 111 in [1] contains several proofs of this. Of course, if p is a prime, F_p may very well be composite; the first example of this is $F_{19} = 4181 = 37 \cdot 113$. Here is the list of the next few terms F_p that are composite:

$$\begin{aligned} F_{31} &= 1346269 = 557 \cdot 2471, \\ F_{37} &= 24157817 = 73 \cdot 149 \cdot 2221, \\ F_{41} &= 165580141 = 2789 \cdot 59369, \\ F_{53} &= 53316291173 = 953 \cdot 55945741. \end{aligned}$$

In this note we show that in fact F_p is composite for certain primes p. We prove the following result.

Theorem: Let $p > 7$ be a prime satisfying the following two conditions:

I. $p \equiv 2 \pmod 5$ or $p \equiv 4 \pmod 5$;

II. $2p - 1$ is also a prime.

Then F_p is composite, in fact, $(2p-1) \mid F_p$.

Proof: We start with the explicit formula for F_p:

$$F_p = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^p - \left(\frac{1-\sqrt{5}}{2} \right)^p \right\}.$$

Multiplying out by $\sqrt{5}$ and squaring, we get

$$5F_p^2 = \frac{1}{2^{2p}} \left\{ \left(1+\sqrt{5}\right)^{2p} + \left(1-\sqrt{5}\right)^{2p} \right\} + 2$$

or

$$2^{2p-1} \cdot 5F_p^2 = \frac{1}{2} \left\{ \left(1+\sqrt{5}\right)^{2p} + \left(1-\sqrt{5}\right)^{2p} \right\} + 2^{2p}.$$

When we expand the powers inside the braces, the terms involving $\sqrt{5}$ will cancel out and we get

$$2^{2p-1} \cdot 5F_p^2 = 1 + \binom{2p}{2} 5 + \binom{2p}{4} 5^2 + \cdots + \binom{2p}{2p-2} 5^{p-1} + 5^p + 2^{2p}.$$

Since $2p-1$ is a prime, $2^{2p-1} \equiv 2 \pmod{2p-1}$ and $\binom{2p}{k} \equiv 0 \pmod{2p-1}$ for $k < 2p-1$, so

$$5F_p^2 \cdot 2 \equiv 1 + 5^p + 4 \pmod{2p-1}$$

or

$$2F_p^2 \equiv 5^{p-1} + 1 \pmod{2p-1}. \tag{1}$$

Now let $\left(\frac{a}{b}\right)$ denote the Legendre symbol. By Euler's theorem,

$$5^{p-1} \equiv \left(\frac{5}{2p-1}\right) \pmod{2p-1}. \tag{2}$$

Suppose $p \equiv 2 \pmod 5$ so that $p = 5k+2$ for some integer k. Since $2p-1$ is a prime, by Gauss's reciprocity theorem,

$$\left(\frac{5}{2p-1}\right)\left(\frac{2p-1}{5}\right) = (-1)^{\frac{4}{2}\cdot\frac{2p-2}{2}} = 1$$

so that

$$\left(\frac{5}{2p-1}\right) = \left(\frac{2p-1}{5}\right) = \left(\frac{10k+3}{5}\right) = \left(\frac{3}{5}\right) = -1.$$

Hence, by (1) and (2),

$$2F_p^2 \equiv -1+1 = 0 \pmod{2p-1}.$$

This means that $2p-1$ divides F_p, and since $F_p > 2p-1$ for $p > 7$, F_p is composite.

In a similar way, if $p \equiv 4 \pmod 5$, $p = 5k+4$ for some k, and

$$\left(\frac{5}{2p-1}\right) = \left(\frac{2p-1}{5}\right) = \left(\frac{10k+7}{5}\right) = \left(\frac{2}{5}\right) = -1,$$

and again, as before, $2p-1$ divides F_p.

Here is the list of the first 21 prime indices p for which the above Theorem guarantees F_p to be composite: 19, 37, 79, 97, 139,157, 199, 229, 307, 337, 367, 379, 439, 499, 547, 577, 607, 619, 727, 829, 839.

REFERENCE

1. M. Bicknell & V. E. Hoggatt. *A Primer for the Fibonacci Numbers.* Santa Clara, Calif.: The Fibonacci Association, 1973.

AMS Classification Numbers: 11A51, 11B39

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