ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to stanley@tiac.net *on the Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-889 ***Proposed by Mario DeNobili, Vaduz, Lichtenstein***

Find 17 consecutive Fibonacci numbers whose average is a Lucas number.

B-890 ***Proposed by Stanley Rabinowitz, Westford, MA***

If $F_{-a}F_bF_{a-b} + F_{-b}F_cF_{b-c} + F_{-c}F_aF_{c-a} = 0$, show that either $a = b$, $b = c$, or $c = a$.

B-891 ***Proposed by Aloysius Dorp, Brooklyn, NY***

Let $\langle P_n \rangle$ be the Pell numbers defined by $P_0 = 0$, $P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$ for $n \geq 0$. Find integers a, b, and m such that $L_n \equiv P_{an+b} \pmod{m}$ for all integers n.

B-892 ***Proposed by Stanley Rabinowitz, Westford, MA***

Show that, modulo 47, $F_n^2 - 1$ is a perfect square if n is not divisible by 16.

B-893 ***Proposed by Aloysius Dorp, Brooklyn, NY***

Find integers a, b, c, and d so that

$$F_xF_yF_z + aF_{x+1}F_{y+1}F_{z+1} + bF_{x+2}F_{y+2}F_{z+2} + cF_{x+3}F_{y+3}F_{z+3} + dF_{x+4}F_{y+4}F_{z+4} = 0$$

is true for all x, y, and z.

B-894 ***Proposed by the editor***

Solve for x:

$$F_{110}^x + 442F_{115}^x + 13F_{119}^x = 221F_{114}^x + 255F_{117}^x.$$

SOLUTIONS

Absolute Sum

B-871 ***Proposed by Paul S. Bruckman, Berkeley, CA***
(Vol. 37, no. 1, February 1999)

Prove that

$$\sum_{k=0}^{2n}\binom{2n}{k}|n-k|^3 = n^2\binom{2n}{n}.$$

Solution by Indulis Strazdins, Riga Technical University, Latvia

The sum is equal to

$$S(n) = 2\sum_{k=0}^{n-1}(n-k)^3\binom{2n}{k} = 2n^3 s_0 - 6n^2 s_1 + 6ns_2 - 2s_3,$$

where the expressions

$$s_m = \sum_{k=0}^{n-1}k^m\binom{2n}{k} \quad (m = 0, 1, 2, 3)$$

can be derived from the known formulas

$$\sum_{k=0}^{n}\binom{n}{k} = 2^n,$$

$$\sum_{k=0}^{n}k\binom{n}{k} = n\cdot 2^{n-1},$$

$$\sum_{k=0}^{n}k^2\binom{n}{k} = n(n+1)\cdot 2^{n-2},$$

$$\sum_{k=0}^{n}k^3\binom{n}{k} = n^2(n+3)\cdot 2^{n-3}.$$

The results are

$$s_0 = 2^{2n-1} - \frac{1}{2}\binom{2n}{n},$$

$$s_1 = n\left(2^{2n-1} - \binom{2n}{n}\right),$$

$$s_2 = n\left((2n+1)2^{2n-2} - \frac{3}{2}n\binom{2n}{n}\right),$$

$$s_3 = n^2\left((2n+3)2^{2n-2} - \frac{1}{2}(4n+1)\binom{2n}{n}\right).$$

Thus,

$$S(n) = (4n^3 - 12n^3 + 6n^2(2n+1) - 2n^2(2n+3))2^{2n-2} - (n^3 - 6n^3 + 9n^3 - n^2(4n+1))\binom{2n}{n} = n^2\binom{2n}{n}.$$

Bruckman noted that

$$\sum_{k=0}^{2n}\binom{2n}{k}|n-k| = n\binom{2n}{n}$$

and conjectures that

$$\sum_{k=0}^{2n}\binom{2n}{k}|n-k|^{2r-1} = P_r(n)\binom{2n}{n}$$

for some monic polynomial $P_r(n)$ of degree r.

Solutions also received by H.-J. Seiffert and the proposer.

Rational Recurrence

B-872 ***Proposed by Murray S. Klamkin, University of Alberta, Canada (Vol. 37, no. 2, May 1999)***

Let $r_n = F_{n+1}/F_n$ for $n > 0$. Find a recurrence for $t_n = r_n^2$.

Solution 1 by Maitland A. Rose, University of South Carolina, Sumter, SC

$$t_n = \frac{F_{n+1}^2}{F_n^2} = \frac{F_n^2 + 2F_nF_{n-1} + F_{n-1}^2}{F_n^2} = 1 + \frac{2F_{n-1}}{F_n} + \frac{F_{n-1}^2}{F_n^2} = 1 + \frac{2}{\sqrt{t_{n-1}}} + \frac{1}{t_{n-1}}.$$

Solution 2 by Kathleen E. Lewis, SUNY, Oswego, NY

The identity $F_{n+1}^2 = 2F_n^2 + 2F_{n-1}^2 - F_{n-2}^2$ is straightforward to prove. Dividing by F_n^2 gives

$$t_n = 2 + \frac{2}{t_{n-1}} - \frac{1}{t_{n-1}t_{n-2}}.$$

Klamkin, Morrison, and Seiffert all found the corresponding recurrence for an arbitrary second-order linear recurrence $w_{n+2} = Pw_{n+1} - Qw_n$. If $t_n = (w_{n+1}/w_n)^2$, then

$$t_n = (P^2 - Q) - \frac{(P^2-Q)Q}{t_{n-1}} + \frac{Q^3}{t_{n-1}t_{n-2}}.$$

Solutions also received by Brian D. Beasley, Paul S. Bruckman, Leonard A. G. Dresel, John F. Morrison, Jaroslav Seibert, H.-J. Seiffert, and the proposer.

A Property of 3

B-873 ***Proposed by Herta Freitag, Roanoke, VA (Vol. 37, no. 2, May 1999)***

Prove that 3 is the only positive integer that is both a prime number and of the form $L_{3n} + (-1)^n L_n$.

Solution by L. A. G. Dresel, Readng, England

Put $T_n = L_{3n} + (-1)^n L_n$. Since the Binet forms for L_{3n} and L_n give the identity $L_{3n} = L_n^3 - 3(-1)^n L_n$, we have $T_n = L_n(L_n^2 - 2(-1)^n) = L_n L_{2n}$. Now $L_n = 1$ only if $n = 1$, so that $T_1 = 3$. But when $n \neq 1$, T_n is the product of two integers, each greater than 1. Hence, 3 is the only prime of the form T_n.

Solutions also received by Paul S. Bruckman, Kathleen E. Lewis, John F. Morrison, Jaroslav Seibert, H.-J. Seiffert, Indulis Strazdins, and the proposer.

Another Property of 3

B-874 ***Proposed by David M. Bloom, Brooklyn College, NY***
(Vol. 37, no. 2, May 1999)

Prove that 3 is the only positive integer that is both a Fibonacci number and a Mersenne number. [A Mersenne number is a number of the form 2^a-1.]

Solution by the proposer

If $F_n=2^a-1$ with $a\geq 2$, then $F_n+1=2^a$. But the general identity $F_{a+b}+(-1)^bF_{a-b}=F_aL_b$ shows that

$$\begin{aligned}
n&=4k &&\text{implies}\quad F_n+1=F_{2k-1}L_{2k+1},\\
n&=4k+1 &&\text{implies}\quad F_n+1=F_{2k+1}L_{2k},\\
n&=4k+2 &&\text{implies}\quad F_n+1=F_{2k+2}L_{2k},\\
n&=4k+3 &&\text{implies}\quad F_n+1=F_{2k+1}L_{2k+2}.
\end{aligned}$$

Thus, if $F_n+1=2^a$, the L-factor on the right must be a power of 2. But it must also be less than or equal to 4 since no Lucas number is divisible by 8. Thus, in all cases, $L_{2k}\leq 4$ and $k\geq 1$ since $F_n\geq 3$. Hence, $k=1$ and the result follows.

Solutions also received by Paul S. Bruckman, Leonard A. G. Dresel, and H.-J. Seiffert.

A Third Property of 3

B-875 ***Proposed by Richard André-Jeannin, Cosnes et Romain, France***
(Vol. 37, no. 2, May 1999)

Prove that 3 is the only positive integer that is both a triangular number and a Fermat number. [A triangular number is a number of the form $n(n+1)/2$. A Fermat number is a number of the form 2^a+1.]

Solution by H.-J. Seiffert, Berlin

Let n be a positive integer and a a nonnegative integer such that $n(n+1)/2=2^a+1$. Multiplying by 2 and then subtracting 2 on both sides yields $(n-1)(n+2)=2^{a+1}$. Hence, $n\geq 2$, and $n-1$ and $n+2$ both must be powers of 2. Since $n-1$ and $n+2$ are of opposite parity, we then must have $n-1=2^0$ or $n=2$. This gives $n(n+1)/2=3=2^1+1$.

Solutions also received by Paul S. Bruckman, Leonard A. G. Dresel, Jaroslav Seibert, and the proposer.

Trigonometric Sum

B-876 ***Proposed by N. Gauthier, Royal Military College of Canada***
(Vol. 37, no. 2, May 1999)

Evaluate

$$\sum_{k=1}^{n}\sin\left(\frac{\pi F_{k-1}}{F_kF_{k+1}}\right)\sin\left(\frac{\pi F_{k+2}}{F_kF_{k+1}}\right).$$

Solution by Jaroslav Seibert, University of Education, Czech Republic

For all real numbers x and y, we have

$$\sin\frac{x+y}{2}\sin\frac{x-y}{2}=-\frac{1}{2}(\cos x-\cos y).$$

Therefore,

$$\sum_{k=1}^{n}\sin\left(\frac{\pi F_{k-1}}{F_kF_{k+1}}\right)\sin\left(\frac{\pi F_{k+2}}{F_kF_{k+1}}\right)=\sum_{k=1}^{n}\sin\pi\left(\frac{F_{k+1}-F_k}{F_kF_{k+1}}\right)\sin\pi\left(\frac{F_{k+1}+F_k}{F_kF_{k+1}}\right)$$

$$=-\frac{1}{2}\sum_{k=1}^{n}\left(\cos 2\pi\frac{F_{k+1}}{F_kF_{k+1}}-\cos 2\pi\frac{F_k}{F_kF_{k+1}}\right)$$

$$=-\frac{1}{2}\sum_{k=1}^{n}\left(\cos 2\pi\frac{1}{F_k}-\cos 2\pi\frac{1}{F_{k+1}}\right)$$

$$=-\frac{1}{2}\left(\cos 2\pi\frac{1}{F_1}-\cos 2\pi\frac{1}{F_{n+1}}\right)$$

$$=\frac{1}{2}\left(\cos\frac{2\pi}{F_{n+1}}-1\right)=-\sin^2\frac{\pi}{F_{n+1}}.$$

Solutions also received by Paul S. Bruckman, Charles K. Cook, Mario DeNobili, Leonard A. G. Dresel, John F. Morrison, Maitland A. Rose, H.-J. Seiffert, and the proposer.

Determining the Determinant

B-877 ***Proposed by Indulis Strazdins, Riga Technical University, Latvia (Vol. 37, no. 2, May 1999)***

Evaluate

$$\begin{vmatrix} F_nF_{n+1} & F_{n+1}F_{n+2} & F_{n+2}F_{n+3} & F_{n+3}F_{n+4} \\ F_{n+4}F_{n+5} & F_{n+5}F_{n+6} & F_{n+6}F_{n+7} & F_{n+7}F_{n+8} \\ F_{n+8}F_{n+9} & F_{n+9}F_{n+10} & F_{n+10}F_{n+11} & F_{n+11}F_{n+12} \\ F_{n+12}F_{n+13} & F_{n+13}F_{n+14} & F_{n+14}F_{n+15} & F_{n+15}F_{n+16} \end{vmatrix}.$$

Solution by the proposer

Let $P_n=F_nF_{n+1}$. It is straightforward to prove the identity

$$P_{n+3}=2P_{n+2}+2P_{n+1}-P_n.$$

Hence, the 4th column is a linear combination of the first three ones, and therefore the determinant is 0.

Most of the solvers pointed out analogous results for larger determinants. If the determinant contains the product of k Fibonacci numbers, $F_nF_{n+1}\dots F_{n+k-1}$, then the determinant is 0 when the order of the determinant is at least $k+2$.

Solutions also received by Paul S. Bruckman, Leonard A. G. Dresel, Jaroslav Seibert, H.-J. Seiffert, and the proposer.

Harmonic Inequality

B-878 ***Proposed by L. A. G. Dresel, Reading, England***
(Vol. 37, no. 3, August 1999)

Show that, for positive integers n, the harmonic mean of F_n and L_n can be expressed as the ratio of two Fibonacci numbers, and that it is equal to $L_{n-1}+R_n$, where $|R_n| \leq 1$. Find a simple formula for R_n.

Note: If h is the harmonic mean of x and y, then $2/h = 1/x + 1/y$.

Solution by Harris Kwong, SUNY College at Fredonia, NY

The harmonic mean of F_n and L_n is given by

$$\frac{2F_nL_n}{F_n+L_n} = \frac{2F_{2n}}{F_n+F_{n-1}+F_{n+1}} = \frac{F_{2n}}{F_{n+1}} = L_{n-1}+\frac{(-1)^n}{F_{n+1}},$$

in which $F_{2n} = F_{n+1}L_{n-1}+(-1)^n$ follows from Binet's formulas.

Solutions also received by Paul S. Bruckman, Charles K. Cook, Don Redmond, H.-J. Seiffert, James A. Sellers, Indulis Strazdins, and the proposer.

Addenda. We wish to belatedly acknowledge solutions from the following solvers:

Brian Beasley solved B-854, 855, 857, 860, 862, 863, and 864.

L. A. G. Dresel solved B-866, 867, 868, 869, and 870.

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