ON DIOPHANTINE APPROXIMATIONS WITH RATIONALS RESTRICTED BY ARITHMETICAL CONDITIONS

Carsten Elsner
Institut für Mathematik, Universität Hannover
Welfengarten 1, 30167 Hannover, Federal Republic of Germany
(*Submitted March 1998-Final Revision August 1998*)

1. INTRODUCTION AND STATEMENT OF RESULTS

One of the most important applications of continued fractions deals with the approximation of real numbers by rationals. The famous approximation theorem of A. Hurwitz [7] states that for every real irrational number ξ there are infinitely many integers u and $v > 0$ such that

$$\left|\xi-\frac{u}{v}\right| \leq \frac{1}{\sqrt{5}v^2}.$$

The constant $1/\sqrt{5}$ is well known to be best-possible in general.

S. Hartman [6] was the first to introduce congruence conditions on u and v; the best approximation result of this type up until now is due to S. Uchiyama [12]:

For any irrational number ξ, any $s>1$, and integers a and b, there are infinitely many integers u and $v \neq 0$ such that

$$\left|\xi-\frac{u}{v}\right| < \frac{s^2}{4v^2} \tag{1.1}$$

and

$$u \equiv a \bmod s, \quad v \equiv b \bmod s, \tag{1.2}$$

provided that a and b are not both divisible by s.

A weaker theorem was proved by J. F. Koksma [9] in 1951. Recently, the author [2] has shown that the constant $1/4$ in (1.1) is best-possible.

But one expects that weaker arithmetical conditions in (1.2) on numerators and denominators will imply smaller constants in (1.1). A result of this kind is proved in [3]:

Let $0< \varepsilon \leq 1$, and let p be a prime with

$$p > \left(\frac{2}{\varepsilon}\right)^2;$$

h denotes any integer that is not divisible by p. Then, for any real irrational number ξ, there are infinitely many integers u and $v>0$ satisfying

$$\left|\xi-\frac{u}{v}\right| \leq \frac{(1+\varepsilon)p^{3/2}}{\sqrt{5}v^2} \tag{1.3}$$

and

$$u \equiv hv \not\equiv 0 \bmod p. \tag{1.4}$$

In this paper, we shall improve this result as far as possible, where additionally *coprime* integers u and v are considered.

Theorem 1.1: Let s denote any positive integer having an odd prime divisor p such that $p^\alpha \mid s$ for some positive integer α. Moreover, let h be any integer. Then, for every real irrational number ξ, there are infinitely many integers u and $v > 0$ satisfying

$$\left|\xi - \frac{u}{v}\right| \le \frac{s}{\sqrt{5}v^2}$$

and

$$u \equiv hv \mod s, \quad (u, v) \le \frac{s}{p^\alpha}.$$

In general, the constant $1/\sqrt{5}$ is best-possible.

Corollary 1.1: Let $s = p^\alpha$ denote some prime power with an odd prime p. Moreover, let h be any integer. Then, for every real irrational number ξ, there are infinitely many coprime integers u and $v > 0$ satisfying

$$\left|\xi - \frac{u}{v}\right| \le \frac{s}{\sqrt{5}v^2} \tag{1.5}$$

and

$$u \equiv hv \mod s. \tag{1.6}$$

By Theorem 3.2 in [1] with $\delta = 1/10$ and $\xi = 12 + \sqrt{145}$, all fractions u/v with odd coprime integers u and $v > 0$ satisfy

$$\left|\xi - \frac{u}{v}\right| > \frac{2}{\sqrt{5}v^2}.$$

Hence, Corollary 1.1 does not hold in the case $s = 2$ and $h = 1$. Also, the bound on the right of (1.5) must be enlarged in the case of moduli s having more than one prime divisor.

Theorem 1.2: Let s be some positive integer having at least two prime divisors. Moreover, h denotes any integer. Then there is a real quadratic irrational number ξ with the following property. For every pair u and v of coprime integers with $|v| > 1$ and $u \equiv hv \mod s$, the inequality

$$\left|\xi - \frac{u}{v}\right| > \frac{s}{2v^2}$$

holds.

It is suggested by the above-mentioned theorems that approximation results with an additional condition like (1.6) depend on arithmetic properties of the modulus s. A general result of this kind is expressed in our final Theorem 1.3. For an integer $s > 1$, the number

$$\delta(s) := \prod_{p \mid s} p$$

is the square-free kernel of s, where p runs through the prime divisors of s. In what follows, p_0 is the smallest prime divisor of s, and

$$S := \min\left\{\frac{s^2}{4}, \frac{s\delta^2(s)}{p_0^2}\right\}.$$

Theorem 1.3: For arbitrary integers $s > 1$ and h and for every real irrational number ξ, there are infinitely many coprime integers u and $v > 0$ satisfying

$$\left|\xi - \frac{u}{v}\right| < \frac{S}{v^2}$$

and

$$u \equiv hv \mod s.$$

For further improvements of the bound on the right-hand side of (1.5) in Corollary 1.1 for all numbers ξ from a certain set of measure 1, the author [4] applies the mean value theorem of Gauss-Kusmin-Lévy [10] from the metric theory of continued fractions. This set depends on p. To prove our theorems, we shall need some well-known elementary facts from the theory of continued fractions (see [11] or [5]). By

$$\xi = [a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}},$$

we denote the continued fraction expansion of a real number ξ.

2. PROOFS OF THEOREMS 1.1 AND 1.2

Proof of Theorem 1.1: The proof of Theorem 1.1 is based on the following proposition.

Proposition 2.1: Let $p > 2$ be a prime number. Among any six consecutive convergents p_{n+i} / q_{n+i} $(n \geq 0,\ i = 0, 1, 2, 3, 4, 5)$ of a real irrational number η there is at least one fraction, say p_v / q_v, such that

$$\left|\eta - \frac{p_v}{q_v}\right| \leq \frac{1}{\sqrt{5}q_v^2} \tag{2.1}$$

holds and q_v is not divisible by p.

Proof: We denote the set of fractions from $\frac{p_n}{q_n}, \ldots, \frac{p_{n+5}}{q_{n+5}}$ satisfying (2.1) by $\mathcal{A}_n$. From a famous theorem of A. Hurwitz which asserts that at least one of three consecutive convergents satisfies (2.1) (see, e.g., Satz 15, ch. 2 in [11]), we know that $2 \leq |\mathcal{A}_n| \leq 6$. In what follows, we consider several cases according to the distribution of fractions from $\mathcal{A}_n$.

Case 1. There is an integer m such that $\frac{p_m}{q_m}, \frac{p_{m+1}}{q_{m+1}} \in \mathcal{A}_n$.

It is a well-known fact that q_m and q_{m+1} represent coprime integers and, therefore, the prime number p cannot divide both of the numbers q_m and q_{m+1}.

Case 2. There are no consecutive convergents of η in $\mathcal{A}_n$.

Case 2.1. It is $\frac{p_m}{q_m}, \frac{p_{m+2}}{q_{m+2}} \in \mathcal{A}_n$ for some integer m.

Let us assume that p divides both q_m and q_{m+2}. Then the recurrence formula of the q's yields

$$a_{m+2}q_{m+1} = q_{m+2} - q_m \equiv 0 \mod p.$$

From $(q_m, q_{m+1}) = 1$, we know that q_{m+1} is not divisible by p. Therefore, p divides a_{m+2}, and we have $a_{m+2} \geq p > \sqrt{5}$. It follows that

$$\left|\eta - \frac{p_{m+1}}{q_{m+1}}\right| < \frac{1}{a_{m+2}q_{m+1}^2} < \frac{1}{\sqrt{5}q_{m+1}^2},$$

hence $\frac{p_{m+1}}{q_{m+1}} \in \mathcal{A}_n$. But we know that $\frac{p_m}{q_m} \in \mathcal{A}_n$ from the hypothesis of Case 2.1, which is incompatible with the hypothesis of Case 2. We have proved that $p \mid q_m$ and $p \mid q_{m+2}$ cannot hold simultaneously.

Case 2.2. It is $\frac{p_m}{q_m}, \frac{p_{m+3}}{q_{m+3}} \in \mathcal{A}_n$ for some integer m.

As in the preceding case, we assume that p divides both of the denominators q_m and q_{m+3}. We have

$$q_{m+3} = a_{m+3}q_{m+2} + q_{m+1},$$
$$q_{m+2} = a_{m+2}q_{m+1} + q_m,$$

for some positive integers a_{m+2}, a_{m+3} from the continued fraction expansion of η. Putting the second equation into the first one, we obtain the identity

$$q_{m+3} - a_{m+3}q_m = (a_{m+2}a_{m+3} + 1)q_{m+1}.$$

Our assumption on p implies that the integer $(a_{m+2}a_{m+3} + 1)q_{m+1}$ is divisible by p. Since q_m and q_{m+1} are coprime, $p \mid q_{m+1}$ is impossible. It follows that p divides $a_{m+2}a_{m+3} + 1$ and, consequently, we have $a_{m+2}a_{m+3} + 1 \geq p \geq 3$. Hence, it is impossible to have $a_{m+2} = a_{m+3} = 1$. We discuss the remaining cases.

Case 2.2.1. $a_{m+2} \geq 3$ or $a_{m+3} \geq 3$.

From

$$\left|\eta - \frac{p_n}{q_n}\right| < \frac{1}{a_{n+1}q_n^2} \quad (n \geq 1),$$

we get

$$\frac{p_m}{q_m}, \frac{p_{m+1}}{q_{m+1}} \in \mathcal{A}_n \qquad (\text{if } a_{m+2} \geq 3),$$
$$\frac{p_{m+2}}{q_{m+2}}, \frac{p_{m+3}}{q_{m+3}} \in \mathcal{A}_n \quad (\text{if } a_{m+3} \geq 3).$$

Again there is a contradiction to the hypothesis of Case 2.

Case 2.2.2. $a_{m+2} = a_{m+3} = 2$.

We have

$$\alpha_{m+2} := [2; 2, a_{m+4}, a_{m+5}, \dots] > 2 + \frac{1}{2+1} = \frac{7}{3},$$

and finally, it follows that

$$\left|\eta - \frac{p_{m+1}}{q_{m+1}}\right| < \frac{1}{\alpha_{m+2}q_{m+1}^2} < \frac{3}{7q_{m+1}^2} < \frac{1}{\sqrt{5}q_{m+1}^2}.$$

Hence, it is

$$\frac{p_m}{q_m}, \frac{p_{m+1}}{q_{m+1}} \in \mathcal{A}_n,$$

a contradiction.

Case 2.2.3. $a_{m+2} = 2,\ a_{m+3} = 1$.

It is

$$\alpha_{m+2} := [2; 1, a_{m+4}, a_{m+5}, \ldots] > 2 + \frac{1}{1+1} = \frac{5}{2}$$

and

$$\left|\eta - \frac{p_{m+1}}{q_{m+1}}\right| < \frac{2}{5q_{m+1}^2} < \frac{1}{\sqrt{5}q_{m+1}^2}.$$

Again we get

$$\frac{p_m}{q_m}, \frac{p_{m+1}}{q_{m+1}} \in \mathcal{A}_n.$$

Case 2.2.4. $a_{m+2} = 1,\ a_{m+3} = 2$.

First, note that $\alpha_{m+3} := [2; a_{m+4}, a_{m+5}, \ldots] > 2$. We get .

$$\left|\eta - \frac{p_{m+2}}{q_{m+2}}\right| = \frac{1}{q_{m+2}(\alpha_{m+3}q_{m+2} + q_{m+1})} < \frac{1}{q_{m+2}^2\left(2 + \frac{q_{m+1}}{q_{m+2}}\right)}$$

$$= \frac{1}{q_{m+2}^2\left(2 + \frac{1}{[1; a_{m+1}, \ldots, a_1]}\right)} < \frac{2}{5q_{m+2}^2}$$

by $[1; a_{m+1}, \ldots, a_1] < 2$. The contradiction arises from

$$\frac{p_{m+2}}{q_{m+2}}, \frac{p_{m+3}}{q_{m+3}} \in \mathcal{A}_n.$$

Hence, it is proved that $p \mid q_m$ and $p \mid q_{m+3}$ cannot hold simultaneously. Since for every integer $m \geq 0$ there is at least one fraction among the convergents $\frac{p_{m+1}}{q_{m+1}}$, $\frac{p_{m+2}}{q_{m+2}}$, and $\frac{p_{m+3}}{q_{m+3}}$ satisfying (2.1) by Hurwitz's theorem, we have finished the proof of Proposition 2.1.

By the hypotheses of Theorem 1.1 on ξ, h, and s, we may choose $\eta := (\xi - h)/s$. From Proposition 2.1, we know that there are infinitely many convergents p_m / q_m of η with

$$\left|\frac{\xi - h}{s} - \frac{p_m}{q_m}\right| \leq \frac{1}{\sqrt{5}q_m^2},$$

where p and q_m are coprime integers. Put $u := hq_m + sp_m$ and $v := q_m$. Then, it is $u \equiv hv \bmod s$ and

$$\frac{1}{s}\left|\xi - \frac{u}{v}\right| \leq \frac{1}{\sqrt{5}v^2}.$$

To estimate the greatest common divisor of u and v, we conclude from $(p_m, q_m) = 1$, $p \nmid q_m$, and $p^\alpha \mid s$ that

$$(sp_m, q_m) = (s, q_m) \le \frac{s}{p^\alpha}.$$

By $(u, v) = (hq_m + sp_m, q_m) = (sp_m, q_m)$, the first assertion of Theorem 1.1 follows.

The corresponding assertion of Corollary 1.1 follows immediately. But it remains to show that Theorem 1.1 cannot be improved in general. For this purpose, let $s > 0$ and h be integers. Put $\xi := h + s(1+\sqrt{5})/2$. In what follows, we shall show that for every $\varepsilon > 0$ there are at most finitely many fractions u/v, where $v > 0$,

$$u \equiv hv \mod s \tag{2.2}$$

and

$$\left|\xi - \frac{u}{v}\right| < \frac{(1-\varepsilon)s}{\sqrt{5}v^2}. \tag{2.3}$$

There is nothing to prove in the case in which no fractions u/v satisfy (2.2) and (2.3) simultaneously. Otherwise, we conclude from (2.2) that $u = hv + ws$ holds for a certain integer w. Then we have, by (2.3),

$$\frac{(1-\varepsilon)s}{\sqrt{5}v^2} > s \cdot \left|\frac{1+\sqrt{5}}{2} - \frac{w}{v}\right|,$$

which yields

$$\left|\frac{1+\sqrt{5}}{2} - \frac{w}{v}\right| < \frac{1-\varepsilon}{\sqrt{5}v^2}. \tag{2.4}$$

It is a well-known fact from the theory of continued fractions that there are at most finitely many solutions w/v in (2.4) (see, e.g., Th. 194 in [5]). One knows that every solution of (2.4) satisfies

$$\frac{w}{v} = \frac{F_{n+1}}{F_n} \text{ for some integer } n, \text{ and } v^2 < \frac{1}{5\varepsilon}.$$

Our assertion follows from the inequality $|v\xi - u| < s/\sqrt{5}$, which has at most finitely many solutions for every integer v.

Proof of Theorem 1.2: Let p and q be different primes with $pq \mid s$. Moreover, we define a sequence $(a_n)_{n\ge0}$ of nonnegative integers as follows. Put $a_0 := 0$ and $a_1 := p$. Let a_2 be the unique solution of the congruence

$$a_2 p \equiv -1 \mod q, \tag{2.5}$$

where $1 \le a_2 < q$. Since $(p, q) = 1$, solutions of (2.5) do exist. Finally, put $a_\nu := p$ for $\nu = 3, 5, 7, \ldots$ and $a_\nu := q$ for $\nu = 4, 6, 8, \ldots$. Then we have $q_0 = 1$, $q_1 = p$, $q_2 = a_2 p + 1 \equiv 0 \mod q$. Applying mathematical induction, we conclude that

$$q_\nu \equiv \begin{cases} 0 \mod p, & \text{if } \nu \equiv 1 \mod 2 \\ 0 \mod q, & \text{if } \nu \equiv 0 \mod 2 \end{cases} \quad (\nu \ge 1). \tag{2.6}$$

Obviously, $\eta := [a_0; a_1, a_2, \ldots]$ and $\xi := h + s\eta$ represent real quadratic irrational numbers.

Now we assume that integers u and v do exist such that $|v| > 1$, $u \equiv hv \mod s$, and

$$\left|\xi - \frac{u}{v}\right| < \frac{s}{2v^2}.$$

Hence, there is an integer w such that $u = hv + ws$ and

$$\left| \eta - \frac{w}{v} \right| < \frac{1}{2v^2}.$$

It follows from the elementary theory of continued fractions (e.g., see Th. 184 in [5]) that the fraction w/v satisfies

$$\frac{w}{v} = \frac{p_n}{q_n} \tag{2.7}$$

for some convergent p_n/q_n of η. One may exclude the case where $n = 0$, since otherwise it follows from (2.7) and $q_0 = 1$ that $v \mid w$. The integer w was defined by $ws = u - hv$, hence v divides u. This is a contradiction to the hypothesis on u and v, because we have deduced from $n = 0$ that $(u, v) = |v| > 1$. Therefore, we may assume $n > 0$ in (2.7). By (2.6), either p or q divides q_n. Since p_n and q_n are coprime, (2.7) implies that v is divisible by the same primes that also divide q_n. From $pq \mid s$ and $u \equiv hv \bmod s$, it follows that $(u, v) > 1$, a contradiction. It is proved that the integers u and v cannot exist, and the proof of Theorem 1.2 is complete.

3. PROOF OF THEOREM 1.3

Let a and b be integers with $a > 0, b \neq 0$. η denotes any real irrational number. In what follows, we consider two consecutive convergents $\frac{p_{n-1}}{q_{n-1}}$ and $\frac{p_n}{q_n}$ of η. For every integer $n \geq 1$ satisfying $aq_n + bq_{n-1} \neq 0$, we define

$$\lambda_n := 1 + \frac{a}{b\alpha_{n+1} - a} - \frac{b}{a\beta_n + b}, \tag{3.1}$$

where $\alpha_{n+1} := [a_{n+1}; a_{n+2}, a_{n+3}, \dots]$ and $\beta_n := [a_n; a_{n-1}, a_{n-2}, \dots, a_1]$. From $\alpha_{n+1} \notin \mathbb{Q}$, we have $b\alpha_{n+1} - a \neq 0$; it follows from

$$\beta_n = \frac{q_n}{q_{n-1}} \quad (n \geq 1)$$

and $aq_n + bq_{n-1} \neq 0$ that $a\beta_n + b \neq 0$.

Proposition 3.1: Let $n \geq 1$ and $\gamma := \operatorname{sign}(b\lambda_n)$. Then we have

$$\left| \eta - \frac{ap_n + bp_{n-1}}{aq_n + bq_{n-1}} \right| = \frac{\gamma ab}{\lambda_n (aq_n + bq_{n-1})^2}.$$

This is Proposition 2.1 in [2] apart from different notations concerning α_n, β_n, and η.

At the beginning of the proof of Theorem 1.3, we apply Uchiyama's result mentioned in the Introduction. By (1.1) and (1.2), there are infinitely many integers u_0 and $v_0 \neq 0$ such that

$$\left| \xi - \frac{u_0}{v_0} \right| < \frac{s^2}{4v_0^2} \tag{3.2}$$

and

$$u_0 \equiv h \mod s, \quad v_0 \equiv 1 \mod s. \tag{3.3}$$

Let $d := (u_0, v_0) > 0$. Every common prime divisor p of d and s is a divisor of v_0, too. This is impossible, because $v_0 \equiv 1 \bmod s$. Hence, d and s are coprime, and therefore an integer d_0 exists such that

$$d \cdot d_0 \equiv 1 \bmod s. \tag{3.4}$$

Moreover, there are coprime integers u and $v \neq 0$ satisfying $u_0 = du$ and $v_0 = dv$. Therefore, we have $d_0 u_0 = dd_0 u$ or $u \equiv hd_0 \bmod s$ by (3.3) and (3.4). Similarly, we conclude $v \equiv d_0 \bmod s$. Collecting together, we have proved the existence of infinitely many coprime integers u and $v \neq 0$ with $u \equiv hv \bmod s$ and, by (3.2),

$$\left|\xi - \frac{u}{v}\right| < \frac{s^2}{4v_0^2} \leq \frac{s^2}{4v^2}.$$

If it is $v < 0$, this result is also true for $-u$ and $-v$, and the assertion of the theorem is proved for $S = s^2/4$.

Now let $\eta := \frac{\xi - h}{s} = [a_0; a_1, a_2, \ldots]$, and let $\frac{p_n}{q_n}$ $(n \geq 0)$ denote the convergents of η. In what follows, we assume $n \geq 1$.

Case 1. $(q_{n-1}, s) = 1$.

Put $P_n := p_{n-1}$, $Q_n := q_{n-1}$. Then we have

$$(P_n, Q_n) = 1, \quad (Q_n, s) = 1, \quad \left|\eta - \frac{P_n}{Q_n}\right| < \frac{1}{Q_n^2}. \tag{3.5}$$

Case 2. $(q_{n-1}, s) > 1$ and $\delta(s) \nmid q_{n-1}$.

Let

$$a := \prod_{\substack{p \mid s \\ p \nmid q_{n-1}}} p, \quad P_n := ap_n + p_{n-1}, \quad Q_n := aq_n + q_{n-1}.$$

From the hypothesis of Case 2, we conclude that

$$a > 1. \tag{3.6}$$

By straightforward computations, one gets $q_n P_n - p_n Q_n = (-1)^n$, which implies that

$$(P_n, Q_n) = 1. \tag{3.7}$$

Let p denote any prime divisor of s. If p divides q_{n-1}, we conclude that a is not divisible by p. Moreover, p does not divide q_n because q_n and q_{n-1} are coprime. Finally, we get $p \nmid Q_n$.

Now, let p and q_{n-1} be coprime. Then we have $p \mid a$, and again p does not divide Q_n. Since p is an arbitrary prime divisor of s, we have proved that

$$(Q_n, s) = 1. \tag{3.8}$$

From the hypothesis $(q_{n-1}, s) > 1$, we know that a certain common prime divisor of q_{n-1} and s exists. This and (3.6) imply that

$$1 < a \leq \frac{\delta(s)}{p_0}, \tag{3.9}$$

where p_0 denotes the smallest prime divisor of s. We apply Proposition 3.1 with $b = 1$:

$$\left|\eta-\frac{P_n}{Q_n}\right| = \frac{a}{|\lambda_n|Q_n^2}, \tag{3.10}$$

where

$$\begin{aligned}\frac{1}{|\lambda_n|} &= \left|\frac{a(1+\alpha_{n+1}\beta_n)-a^2\beta_n+\alpha_{n+1}-2a}{a(1+\alpha_{n+1}\beta_n)}\right| \\ &= \left|1-\frac{2+a\beta_n-\frac{\alpha_{n+1}}{a}}{1+\alpha_{n+1}\beta_n}\right| = |1-\rho_n|.\end{aligned} \tag{3.11}$$

We are looking for a suitable upper bound of $|\lambda_n|^{-1}$. For this purpose, we separate the arguments into three cases.

Case 2.1. $2+a\beta_n-\dfrac{\alpha_{n+1}}{a}<0.$

For $n\geq 1$, it is clear that $\alpha_{n+1}>1$ and $\beta_n>1$. It follows from (3.11) that

$$|\lambda_n|^{-1}=1-\rho_n < 1+\frac{\alpha_{n+1}}{a(1+\alpha_{n+1}\beta_n)} < 1+\frac{\alpha_{n+1}}{a(1+\alpha_{n+1})} < 1+\frac{1}{a}<2. \tag{3.12}$$

Case 2.2. $0\leq 2+a\beta_n-\dfrac{\alpha_{n+1}}{a}\leq 1+\alpha_{n+1}\beta_n.$

Then we have $0\leq\rho_n\leq 1$, and consequently

$$|\lambda_n|^{-1}\leq 1. \tag{3.13}$$

Case 2.3. $2+a\beta_n-\dfrac{\alpha_{n+1}}{a}>1+\alpha_{n+1}\beta_n.$

We conclude that

$$|\lambda_n|^{-1}=\rho_n-1<\frac{2+a\beta_n}{1+\alpha_{n+1}\beta_n}-1<\frac{2+a\beta_n}{1+\beta_n}-1<\frac{2}{1+1}+\frac{a\beta_n}{1+\beta_n}-1<a. \tag{3.14}$$

We know that $a\geq 2$, from (3.6). Collecting together from (3.12) through (3.14) we have proved that $|\lambda_n|^{-1}<a$ holds for every integer $n\geq 1$. Hence, (3.10) yields

$$\left|\eta-\frac{P_n}{Q_n}\right| < \frac{a^2}{Q_n^2}. \tag{3.15}$$

Case 3. $\delta(s)\,|\,q_{n-1}.$

Since q_{n-1} and q_n are coprime, it follows from the hypothesis that $(q_n, s)=1$. Put $P_n:=p_n$ and $Q_n:=q_n$. Obviously, the assertions for P_n and Q_n from (3.5) hold.

We collect together the results from (3.5), (3.7), (3.8), and (3.15): For a certain sequence of increasing integers $n\geq 1$, we get a sequence of rationals $(P_\nu/Q_\nu)_{\nu\geq 1}$ with coprime integers P_ν and Q_ν such that $(Q_\nu, s)=1$ $(\nu\geq 1)$,

$$Q_1\leq Q_2\leq Q_3\leq\cdots\leq Q_\nu\to\infty$$

and

$$\left|\frac{\xi-h}{s}-\frac{P_\nu}{Q_\nu}\right|<\frac{a^2}{Q_\nu^2}\quad(\nu\geq 1).$$

Let $u := hQ_v + sP_v$, $v := Q_v$. Then, by the upper bound for a from (3.9), we have

$$\left|\xi - \frac{u}{v}\right| < \frac{s\delta^2(s)}{p_0^2 v^2},$$

where $u \equiv hv \mod s$. We conclude from $(Q_v, sP_v) = 1$ that u and v are coprime. Since Q_v can be chosen as large as possible, the assertion of Theorem 1.3 is also proved for $S = s \cdot \delta^2(s) \cdot p_0^{-2}$.

4. CONCLUDING REMARK

Using the well-known continued fraction expansion of Euler's number e, the author obtained the following result.

Theorem: For every integer $s \geq 2$ there are infinitely many fractions P/Q with coprime integers $P, Q > 0$ satisfying $P \equiv Q \equiv 1 \mod s$ and $Q \cdot |Qe - P| = o(1)$ for $Q \to \infty$.

ACKNOWLEDGMENT

I take this opportunity to thank Professor R. C. Vaughan, who suggested to me the basic concept for improving my former results on this topic of diophantine approximation.

REFERENCES

1. L. C. Eggan. "On Diophantine Approximations." *Trans. Amer. Math. Soc.* **99** (1961):102-17.
2. C. Elsner. "On the Approximation of Irrationals by Rationals." *Math. Nachr.* **189** (1998): 243-56.
3. C. Elsner. "On the Approximation of Irrational Numbers with Rationals Restricted by Congruence Relations." *The Fibonacci Quarterly* **34.1** (1996):18-29.
4. C. Elsner. "A Metric Result Concerning the Approximation of Real Numbers by Continued Fractions." *The Fibonacci Quarterly* **36.4** (1998):290-94.
5. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers*. 5th ed. Oxford: Clarendón Press, 1979.
6. S. Hartman. "Sur une condition supplementaire dans les approximations diophantiques." *Coloq. Math.* **2** (1951):48-51.
7. A. Hurwitz. "Über die angenäherte Darstellung der Irrationalzahlen durch rationale Brüche." *Math. Ann.* **39** (1891):279-84.
8. A. Khintchine. *Kettenbrüche*. Leipzig: B. G. Teubner Verlagsgesellschaft, 1956.
9. J. F. Koksma. "Sur l'approximation des nombres irrationnels sous une condition supplementaire." *Simon Stevin* **28** (1951):199-202.
10. P. Lévy. "Sur les lois de probabilité dont dépendent les quotients complets et incomplets d'une fraction continue." *Bull. Soc. Math. France* **57** (1929):178-94.
11. O. Perron. *Die Lehre von den Kettenbrüchen*. New York: Chelsea, 1950.
12. S. Uchiyama. "On Rational Approximations to Irrational Numbers." *Tsukuba J. Math.* **4** (1980):1-7.

AMS Classification Numbers: 11J04, 11J70