A REFINEMENT OF DE BRUYN'S FORMULAS FOR $\sum a^k k^p$

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1. INTRODUCTION

As is known, various methods have been proposed for finding summation formulas for the so-called arithmetic-geometric progression of the form

$$S_{a,p}(n) := \sum_{k=0}^{n} a^k k^p, \tag{1}$$

where a is a real or complex number with $a \neq 0$ and $a \neq 1$, and n and p are nonnegative integers. For some recent papers, see, e.g., de Bruyn [1], Gauthier [4], and Hsu [5]. The object of this note is to show that de Bruyn's formulas expressed in terms of determinants could be given concise explicit forms in terms of Eulerian polynomials. In fact, it is found that the recurrence relations (recursive equations) obtained by de Bruyn for those determinants used in his formulas can be solved by means of Eulerian polynomials.

Let us recall de Bruyn's work briefly. De Bruyn made use of Cramer's rule to develop some explicit formulas for expressing $S_{a,p}(n)$ as $(p+1)\times(p+1)$ determinants. He then gave two formulas for $S_{a,p}(n)$, one in powers of $(n+1)$, the other in powers of n, in which all the coefficients are also expressed as determinants. More precisely, de Bruyn's first formula in powers of $(n+1)$ takes the form

$$S_{a,p}(n) = \frac{a^{n+1}}{a-1}\sum_{r=0}^{p-1}\binom{p}{r} f_r(a)(n+1)^{p-r} + f_p(a)\left(\frac{a^{n+1}-1}{a-1}\right), \tag{2}$$

where $f_p(a) = 1$, and $f_r(a)$ $(r = 1, 2, \ldots, p-1)$ are given by

$$f_r(a) = r!\left(\frac{a}{1-a}\right)^r \det\begin{pmatrix} \frac{1}{1!} & \frac{a-1}{a} & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{2!} & \frac{1}{1!} & \frac{a-1}{a} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & \cdots & \cdots & \cdots & \frac{1}{1!} & \frac{a-1}{a} \\ \frac{1}{r!} & \frac{1}{(r-1)!} & \cdots & \cdots & \cdots & \frac{1}{2!} & \frac{1}{1!} \end{pmatrix}, \tag{3}$$

and $f_0(a), f_1(a), f_2(a), \ldots$ satisfy the recurrence relations

$$f_0(a) = 1,\quad a\sum_{j=0}^{r}\binom{r}{j} f_j(a) - f_r(a) = 0,\quad (r = 1, 2, \ldots). \tag{4}$$

De Bruyn observed that if the f_j's are denoted as the Bernoulli numbers B_j, and we put $a = 1$, equation (4) just gives the well-known recurrence formula for the Bernoulli numbers. This led him to call the numbers $f_r(a)$ $(r = 0, 1, 2, \ldots)$ the a-Bernoulli numbers. In the next section, we shall show that $f_r(a)$ are closely related to Eulerian polynomials.

2. SOLUTION OF RECURSIVE EQUATIONS

Evidently the system of equations given by (4) determines $f_r(a)$'s uniquely with $f_0(a) = 1$. Using (4) recursively one may write

$$f_0(a) = \frac{1}{(1-a)^0},\ f_1(a) = \frac{a}{(1-a)^1},\ f_2(a) = \frac{a+a^2}{(1-a)^2},\ f_3(a) = \frac{a+4a^2+a^3}{(1-a)^3},\ \text{etc.}$$

Here it may be verified that the numerators of the $f_r(a)$'s $(r = 0, 1, 2, \ldots)$ are precisely the Eulerian polynomials, $A_r(a)$ $(r = 0, 1, 2, \ldots)$. In fact, it is known that (cf. Comtet [3], § 6.5)

$$A_0(a) = 1,\ A_1(a) = a,\ A_2(a) = a + a^2,\ A_3(a) = a + 4a^2 + a^3,\ \text{etc.}$$

Thus, one may reasonably conjecture that

$$f_r(a) = \frac{A_r(a)}{(1-a)^r}\ (r = 0, 1, 2, \ldots) \tag{5}$$

are the solutions to the recursive equations given by (4). We will prove this below as a lemma.

The historical origin of Eulerian polynomials $A_p(a)$ is the following summation formula for the infinite arithmetic-geometric series

$$\sum_{k=0}^{\infty} a^k k^p = \frac{A_p(a)}{(1-a)^{p+1}},\ |a| < 1,\ a \neq 0, \tag{6}$$

where $A_p(a)$ is a polynomial of degree p in a, $p \geq 0$, and $0^0 := 1$ (see, e.g., Carlitz [2] and Comtet [3; p. 245]). We shall utilize (6) to prove our preceding conjecture given in the following lemma.

Lemma: The functions $f_r(a)$ given by (5) satisfy the recursive equations displayed in (4) for all complex numbers $a \neq 0, 1$.

Proof: Since $A_0(a) = 1 = f_0(a)$, it suffices to consider equation (4) for $r \geq 1$. Clearly these equations may be equivalently replaced by the following:

$$a\sum_{j=0}^{r}\binom{r}{j}\frac{f_j(a)}{1-a} - \frac{f_r(a)}{1-a} = 0\ (r = 1, 2, 3, \ldots). \tag{7}$$

Substituting (5) into (7) and using the representation (6) for $A_j(a)/(1-a)^{j+1}$ with $|a| < 1$, it is easily found that the left-hand side (LHS) of (7) becomes

$$\begin{aligned} a\sum_{j=0}^{r}\binom{r}{j}\sum_{k=0}^{\infty}a^k k^j - \sum_{k=0}^{\infty}a^k k^r &= \sum_{k=0}^{\infty}a^{k+1}\sum_{j=0}^{r}\binom{r}{j}k^j - \sum_{k=1}^{\infty}a^k k^r\ (r \geq 1) \\ &= \sum_{k=0}^{\infty}a^{k+1}(k+1)^r - \sum_{k=1}^{\infty}a^k k^r = 0. \end{aligned}$$

This shows that (7) holds for the $f_j(a)$'s given by (5) with $|a| < 1$, $a \neq 0$. Now the LHS of (7) [with $f_j(a)$ given by (5)] is a rational function of a that vanishes for infinitely many values of a; thus, it should vanish identically with the only restrictions $a \neq 0, a \neq 1$. This completes the proof of the Lemma.

3. REFINEMENT OF FORMULA (2)

It is known that the Eulerian polynomial $A_r(a)$ $(r \geq 1)$ may be written in the form (cf. Comtet [3; §6.5])

$$A_r(a) = \sum_{k=1}^{r} A(r,k)a^k, \tag{8}$$

where $A(r, k)$ are called Eulerian numbers given explicitly by

$$A(r,k) = \sum_{j=0}^{k} (-1)^j \binom{r+1}{j}(k-j)^r \quad (1 \leq k \leq r). \tag{9}$$

Using the Lemma, one can express de Bruyn 's formula (2) in a refined form. This is given by the following theorem.

Theorem: For any given integer $p \geq 0$, there holds the summation formula

$$S_{a,p}(n) = \frac{1}{a-1}\left[a^{n+1}\sum_{r=0}^{p}\binom{p}{r}\frac{A_r(a)}{(1-a)^r}(n+1)^{p-r} - \frac{A_p(a)}{(1-a)^p}\right], \tag{10}$$

where $A_r(a)$ are given by (8) and (9), $a \neq 0, a \neq 1$.

Remark: De Bruyn's second formula for $S_{a,p}(n)$ in powers of n given by

$$S_{a,p}(n) = \frac{a^{n+1}}{a-1}n^p + \frac{a^n}{a-1}\sum_{r=1}^{p-1}\binom{p}{r}f_r(a)n^{p-r} + f_p(a)\left(\frac{a^n-1}{a-1}\right), \; p > 1,$$

can likewise be refined to the form

$$S_{a,p}(n) = \frac{a^{n+1}}{a-1}n^p + \frac{1}{a-1}\left[a^n\sum_{r=1}^{p}\binom{p}{r}\frac{A_r(a)}{(1-a)^r}n^{p-r} - \frac{A_p(a)}{(1-a)^p}\right]. \tag{11}$$

This is obtained by means of the Lemma. Surely, both (10) and (11) are useful for practical computations whenever n is much larger than p, say $n >> p^3$. Moreover, it may be worth mentioning that the sum $S_{a,p}(n)$ can also be expressed using Stirling numbers of the second kind, and the formula is also available for $n >> p^3$ (cf. [5]).

4. A DIRECT PROOF OF THE THEOREM

Here we shall give a direct computational proof of (10) with the aid of (6). Since (10) is obvious for $p = 0$, it suffices to consider the case $p \geq 1$.

For a given real or complex number a with $a \neq 1$, $a \neq 0$, we shall make use of the simple exponential function ae^θ, θ real or complex. Since $ae^\theta \to a \neq 1$ as $\theta \to 0$, we can find a sufficiently small positive number δ such that $ae^\theta \neq 1$ for $|\theta| < \delta$.

Let us consider the sum

$$S(n,\theta) := \sum_{k=0}^{n}(ae^\theta)^k = \frac{1-a^{n+1}e^{(n+1)\theta}}{1-ae^\theta}, \; (|\theta| < \delta).$$

For given $p \geq 1$, we have the p^{th} derivative with respect to θ:

$$\left(\frac{d^p}{d\theta^p}\right)S(n,\theta)=\sum_{k=1}^{n}a^k k^p e^{k\theta}.$$

Thus, it follows that

$$\begin{aligned} S_{a,p}(n)&=\sum_{k=1}^{n}a^k k^p=\left(\frac{d^p}{d\theta^p}\right)_0 S(n,\theta)\\ &=\left(\frac{d^p}{d\theta^p}\right)_0\left[(1-a^{n+1}e^{(n+1)\theta})(1-ae^{\theta})^{-1}\right], \end{aligned}\tag{12}$$

where the derivatives are evaluated at $\theta=0$. Using Leibniz's product formula for differentiation, we easily find that the RHS of (12) equals

$$\sum_{r=0}^{p-1}\binom{p}{r}(-a^{n+1})(n+1)^{p-r}\left(\frac{d^r}{d\theta^r}\right)_0(1-ae^{\theta})^{-1}+(1-a^{n+1})\left(\frac{d^p}{d\theta^p}\right)_0(1-ae^{\theta})^{-1}.\tag{13}$$

It remains to compute

$$\left(\frac{d^r}{d\theta^r}\right)_0(1-ae^{\theta})^{-1},\ (0\le r\le p).$$

This can be done easily by using (6) with $|a|<1$, $a\neq 0$, as follows:

$$\left(\frac{d^r}{d\theta^r}\right)_0(1-ae^{\theta})^{-1}=\left(\frac{d^r}{d\theta^r}\right)_0\left(\sum_{k=0}^{\infty}a^k e^{k\theta}\right)=\sum_{k=0}^{\infty}a^k k^r=\frac{A_r(a)}{(1-a)^{r+1}}.\tag{14}$$

Here it may be noted that the series $\sum_{k=0}^{\infty}a^k e^{k\theta}$ in (14) can be term-wise differentiated any number of times in a neighborhood of $\theta=0$, say $|\theta|<\delta$, provided that δ is sufficiently small such that $|ae^{\theta}|<\rho=constant<1$ for $|\theta|<\delta$, which obviously implies the uniform convergence condition for the related series.

Now, recalling (12) and substituting (14) into (13), we obtain

$$S_{a,p}(n)=\frac{1}{a-1}\left[a^{n+1}\sum_{r=0}^{p-1}\binom{p}{r}\frac{A_r(a)}{(1-a)^r}(n+1)^{p-r}+(a^{n+1}-1)\frac{A_p(a)}{(1-a)^p}\right].\tag{15}$$

This is precisely equivalent to (10).

Finally, note that (15) is an equality between rational functions of a, valid for infinitely many values of a ($|a|<1$, $a\neq 0$) so that it must be an identity valid for all values of a with the only restrictions $a\neq 1$, $a\neq 0$. This completes the proof of (10).

5. AN EXAMPLE

Consider a pair of trigonometric sums as follows:

$$c(n)=\sum_{k=0}^{n}\alpha^k k^p\cos k\theta,\quad s(n)=\sum_{k=0}^{n}\alpha^k k^p\sin k\theta,$$

where α is a positive real number, $\alpha\neq 1$, p a positive integer, and θ a real number, $0<\theta<2\pi$. These sums can be computed precisely using the explicit formulas (10) or (11). Indeed, taking $a=\alpha e^{i\theta}$ ($i^2=-1$) in (1), we have

$$\sum_{k=0}^{n}(\alpha^k e^{ik\theta})k^p = c(n)+is(n).$$

Denoting the RHS of (10) or of (11) by $\Phi(a, p, n)$, we get

$$c(n) = \operatorname{Re}\Phi(\alpha e^{i\theta}, p, n), \quad s(n) = \operatorname{Im}\Phi(\alpha e^{i\theta}, p, n),$$

where $\operatorname{Re}\Phi$ and $\operatorname{Im}\Phi$ denote the real part and imaginary part of Φ, respectively. Obviously, this follows from the fact that $(\alpha e^{i\theta})^k = \alpha^k \cos k\theta + i\alpha^k \sin k\theta$.

REFERENCES

1. L. Carlitz. "Eulerian Numbers and Polynomials." *Math. Magazine* **32** (1959):247-60.
2. L. Comtet. *Advanced Combinatorics*, Chapter 6, §6.5. Dordrecht: Reidel, 1974.
3. G. F. C. de Bruyn. "Formulas for $a+a^2 2^p + a^3 3^p + \cdots + a^n n^p$." *The Fibonacci Quarterly* **33.2** (1995):98-103.
4. N. Gauthier. "Derivation of a Formula for $\sum r^k x^r$." *The Fibonacci Quarterly* **27.5** (1989): 402-08.
5. L. C. Hsu. "On a Kind of Generalized Arithmetic-Geometric Progression." *The Fibonacci Quarterly* **35.1** (1997):62-67.

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