EQUATIONS INVOLVING ARITHMETIC FUNCTIONS OF FIBONACCI AND LUCAS NUMBERS

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For any positive integer k, let $\phi(k)$ and $\sigma(k)$ be the number of positive integers less than or equal to k and relatively prime to k and the sum of divisors of k, respectively.

In [6] we have shown that $\phi(F_n) \geq F_{\phi(n)}$ and that $\sigma(F_n) \leq F_{\sigma(n)}$ and we have also determined all the cases in which the above inequalities become equalities. A more general inequality of this type was proved in [7].

In [8] we have determined all the positive solutions of the equation $\phi(x^m - y^m) = x^n + y^n$ and in [9] we have determined all the integer solutions of the equation $\phi(|x^m + y^m|) = |x^n + y^n|$.

In this paper, we present the following theorem.

Theorem:

(1) The only solutions of the equation

$$\phi(|F_n|) = 2^m, \tag{1}$$

are obtained for $n = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 9$.

(2) The only solutions of the equation

$$\phi(|L_n|) = 2^m, \tag{2}$$

are obtained for $n = 0, \pm 1, \pm 2, \pm 3$.

(3) The only solutions of the equation

$$\sigma(|F_n|) = 2^m, \tag{3}$$

are obtained for $n = \pm 1, \pm 2, \pm 4, \pm 8$.

(4) The only solutions of the equation

$$\sigma(|L_n|) = 2^m, \tag{4}$$

are obtained for $n = \pm 1, \pm 2, \pm 4$.

Let $n \geq 3$ be a positive integer. It is well known that the regular polygon with n sides can be constructed with the ruler and the compass if and only if $\phi(n)$ is a power of 2. Hence, the above theorem has the following immediate corollary.

Corollary:

(1) The only regular polygons that can be constructed with the ruler and the compass and whose number of sides is a Fibonacci number are the ones with 3, 5, 8, and 34 sides, respectively.

(2) The only regular polygons that can be constructed with the ruler and the compass and whose number of sides is a Lucas number are the ones with 3 and 4 sides, respectively.

The question of finding all the regular polygons that can be constructed with the ruler and the compass and whose number of sides n has various special forms has been considered by us previously. For example, in [10] we found all such regular polygons whose number of sides n belongs to the Pascal triangle and in [11] we found all such regular polygons whose number of sides n is a difference of two equal powers.

We begin with the following lemmas.

Lemma 1:

(1) $F_{-n} = (-1)^{n+1}F_n$ and $L_{-n} = (-1)^n L_n$.

(2) $2F_{m+n} = F_m L_n + L_n F_m$ and $2L_{m+n} = 5F_m F_n + L_m L_n$.

(3) $F_{2n} = F_n L_n$ and $L_{2n} = L_n^2 + 2(-1)^{n+1}$.

(4) $L_n^2 - 5F_n^2 = 4(-1)^n$.

Proof: See [2]. □

Lemma 2:

(1) Let $p > 5$ be a prime number. If $\left(\frac{5}{p}\right) = 1$, then $p \mid F_{p-1}$. Otherwise, $p \mid F_{p+1}$.

(2) $(F_m, F_n) = F_{(m,n)}$ for all positive integers m and n.

(3) If $m \mid n$ and n/m is odd, then $L_m \mid L_n$.

(4) Let p and n be positive integers such that p is an odd prime. Then $(L_p, F_n) > 2$ if and only if $p \mid n$ and n/p is even.

Proof: (1) follows from Theorem XXII in [1].

(2) follows either from Theorem VI in [1] or from Theorem 2.5 in [3] or from the Main Theorem in [12].

(3) follows either from Theorem VII in [1] or from Theorem 2.7 in [3] or from the Main Theorem in [12].

(4) follows either from Theorem 2.9 in [3] or from the Main Theorem in [12]. □

Lemma 3: Let $k \geq 3$ be an integer.

(1) The period of $(F_n)_{n\geq 0}$ modulo 2^k is $2^{k-1}\cdot 3$.

(2) $F_{2^{k-2}\cdot 3} \equiv 2^k \pmod{2^{k+1}}$. Moreover, if $F_n \equiv 0 \pmod{2^k}$, then $n \equiv 0 \pmod{2^{k-2}\cdot 3}$.

(3) Assume that n is an odd integer such that $F_n \equiv \pm 1 \pmod{2^k}$. Then $F_n \equiv 1 \pmod{2^k}$ and $n \equiv \pm 1 \pmod{2^{k-1}\cdot 3}$.

Proof: (1) follows from Theorem 5 in [13].

(2) The first congruence is Lemma 1 in [4]. The second assertion follows from Lemma 2 in [5].

(3) We first show that $F_n \not\equiv -1 \pmod{2^k}$. Indeed, by (1) above and the Main Theorem in [4], it follows that the congruence $F_n \equiv -1 \pmod{2^k}$ has only one solution $n \pmod{2^{k-1}\cdot 3}$. Since $F_{-2} = -1$, it follows that $n \equiv -2 \pmod{2^{k-1}\cdot 3}$. This contradicts the fact that n is odd.

We now look at the congruence $F_n \equiv 1 \pmod{2^k}$. By (1) above and the Main Theorem in [4], it follows that this congruence has exactly three solutions $n \pmod{2^{k-1}\cdot 3}$. Since $F_{-1} = F_1 = F_2 = 1$, it follows that $n \equiv \pm 1, 2 \pmod{2^{k-1}\cdot 3}$. Since n is odd, it follows that $n \equiv \pm 1 \pmod{2^{k-1}\cdot 3}$. □

Lemma 4: Let $k \geq 3$ be a positive integer. Then

$$L_{2^k} \equiv \begin{cases} 2^{k+1}3 - 1 \pmod{2^{k+4}} & \text{if } k \equiv 1 \pmod{2}, \\ 2^{k+1}5 - 1 \pmod{2^{k+4}} & \text{if } k \equiv 0 \pmod{2}. \end{cases}$$

Proof: One can check that the asserted congruences hold for $k = 3$ and 4. We proceed by induction on k. Assume that the asserted congruence holds for some $k \geq 3$.

Suppose that k is odd. Then $L_{2^k} = 2^{k+1}3 - 1 + 2^{k+4}l$ for some integer l. Using Lemma 1(3), it follows that

$$L_{2^{k+1}} = L_{2^k}^2 - 2 = ((2^{k+1}3-1)^2 + 2^{k+5}l(2^{k+1}3-1)^2 + 2^{2k+8}l^2) - 2$$
$$\equiv (2^{k+1}3-1)^2 - 2 \pmod{2^{k+5}}.$$

Hence,

$$L_{2^{k+1}} \equiv 2^{2k+2}9 - 2^{k+2}3 + 1 - 2 \equiv 2^{2k+2}9 + 2^{k+2}(-3) - 1 \pmod{2^{k+5}}.$$

Since $k \geq 3$. it follows that $2k+2 \geq k+5$. Moreover, since $-3 \equiv 5 \pmod{2^3}$, the above congruence becomes

$$L_{2^{k+1}} \equiv 2^{k+2}5 - 1 \pmod{2^{k+5}}.$$

The case k even can be dealt with similarly. □

Proof of the Theorem: In what follows, we will always assume that $n \geq 0$.

(1) We first show that if $\phi(F_n) = 2^m$, then the only prime divisors of n are among the elements of the set $\{2, 3, 5\}$. Indeed, assume that this is not the case. Let $p > 5$ be a prime number dividing n. Since $F_p \mid F_n$, it follows that $\phi(F_p) \mid \phi(F_n) = 2^m$. Hence, $\phi(F_p) = 2^{m_1}$. It follows that

$$F_p = 2^l p_1 \cdot \dots \cdot p_k, \tag{5}$$

where $l > 0$, $k > 0$, and $p_1 < p_2 < \dots < p_k$ are Fermat primes.

Notice that $l = 0$ and $p_1 > 5$. Indeed, since $p > 5$ is a prime, it follows, by Lemma 2(2), that F_p is coprime to F_m for $1 < m \leq 5$. Since $F_3 = 2, F_4 = 3$, and $F_5 = 5$, it follows that $l = 0$ and $p_1 > 5$.

Hence, $p_1 > 5$ for all $i = 1, \dots, k$. Write $p_i = 2^{2^{\alpha_i}} +$ for some $\alpha_i \geq 2$. It follows that

$$p_1 = 4^{2^{\alpha_1 - 1}} + 1 \equiv 2 \pmod{5}.$$

Since $\left(\frac{p_1}{5}\right) = \left(\frac{2}{5}\right) = -1$, it follows, by the quadratic reciprocity law, that $\left(\frac{5}{p_1}\right) = -1$. It follows, by Lemma 2(1), that $p_1 \mid F_{p_1+1}$. Hence,

$$p_1 \mid (F_p, F_{p_1+1}) = F_{(p, p_1+1)}.$$

The above divisibility relation and the fact that p is prime, forces $p \mid p_1 + 1 = 2(2^{2^{\alpha_1}-1} + 1)$. Hence, $p \mid 2^{2^{\alpha_1}-1} + 1$. Thus,

$$p \leq 2^{2^{\alpha_1}-1} + 1. \tag{6}$$

On the other hand, since

$$F_p = \prod_{i=1}^{k} (2^{2^{\alpha_i}} + 1) \equiv 1 \pmod{2^{2^{\alpha_1}}},$$

it follows, by Lemma 3(3), that $p \equiv \pm 1 \pmod{2^{2^{\alpha_1}-1}3}$. In particular,

$$p \geq 2^{2^{\alpha_1}-1}3 - 1. \tag{7}$$

From inequalities (6) and (7), it follows that $2^{2^{\alpha_1}-1}3 - 1 \leq 2^{2^{\alpha_1}-1} + 1$ or $2^{2^{\alpha_1}} \leq 2$. This implies that $\alpha_1 = 0$ which contradicts the fact that $\alpha_1 \geq 2$.

Now write $n = 2^a 3^b 5^c$. We show that $a \leq 2$. Indeed, if $a \geq 3$, then $21 = F_8 \mid F_n$, therefore

$$3 \mid 12 = \phi(21) \mid \phi(F_n) = 2^m,$$

which is a contradiction. We show that $b \leq 2$. Indeed, if $b \geq 3$, then $53 \mid F_{27} \mid F_n$, therefore

$$13 \mid 52 = \phi(53) \mid \phi(F_n) = 2^m,$$

which is a contradiction. Finally, we show that $c \leq 1$. Indeed, if $c \geq 2$, then $3001 \mid F_{25} \mid F_n$, therefore

$$3 \mid 3000 = \phi(3001) \mid \phi(F_n) = 2^m,$$

which is again a contradiction. In conclusion, $n \mid 2^2 \cdot 3^2 \cdot 5 = 180$. One may easily check that the only divisors n of 180 for which $\phi(F_n)$ is a power of 2 are indeed the announced ones.

(2) Since $\phi(2) = \phi(1) = 1 = 2^0$ and $\phi(3) = \phi(4) = 2^1$, it follows that $n = 0, 1, 2, 3$ lead to solutions of equation (2). We now show that these are the only ones. One may easily check that $n \neq 4, 5$. Assume that $n \geq 6$. Since $\phi(L_n) = 2^m$, it follows that

$$L_n = 2^l \cdot p_1 \cdot \dots \cdot p_k, \tag{8}$$

where $l \geq 0$ and $p_1 < \dots < p_k$ are Fermat primes. Write $p_i = 2^{2^{\alpha_i}} + 1$. Clearly, $p_1 \geq 3$. The sequence $(L_n)_{n \geq 0}$ is periodic modulo 8 with period 12. Moreover, analyzing the terms L_s for $s = 0, 1, \dots, 11$, one notices that $L_s \not\equiv 0 \pmod 8$ for any $s = 0, 1, \dots, 11$. It follows that $l \leq 2$ in equation (8). Since $n \geq 6$, it follows that $L_n \geq 18$. In particular, $p_i \geq 5$ for some $i = 1, \dots, k$. From the equation

$$L_n^2 - 5F_n^2 = (-1)^n \cdot 4, \tag{9}$$

it follows easily that $5 \nmid L_n$. Thus, $p_i > 5$. Hence, $p_i = 2^{2^{\alpha_i}} + 1$ for some $\alpha_i \geq 2$. It follows that $p_i \equiv 1 \pmod 4$ and

$$p_i \equiv 4^{2^{\alpha_i - 1}} + 1 \equiv (-1)^{2^{\alpha_i - 1}} + 1 \equiv 2 \pmod 5.$$

In particular, $\left(\frac{p_i}{5}\right) = \left(\frac{2}{5}\right) = -1$. Hence, by the quadratic reciprocity law, it follows that $\left(\frac{5}{p_i}\right) = -1$ as well. On the other hand, reducing equation (9) modulo p_i, it follows that

$$5F_n^2 \equiv (-1)^{n-1} \cdot 4 \pmod{p_i}. \tag{10}$$

Since $p_i \equiv 1 \pmod 4$, it follows that $\left(\frac{(-1)^{n-1}}{p_i}\right) = 1$. From congruence (10), it follows that $\left(\frac{5}{p_i}\right) = 1$, which contradicts the fact that $\left(\frac{5}{p_i}\right) = -1$.

(3) Since $\sigma(1) = 1 = 2^0, \sigma(3) = 4 = 2^2$, and $\sigma(21) = 32 = 2^5$, it follows that $n = 1, 2, 4, 8$ are solutions of equation (3). We show that these are the only ones. One can easily check that $n \neq 3, 5, 6, 7$. Assume now that there exists a solution of equation (3) with $n > 8$. Since $\sigma(F_n) = 2^m$, it follows easily that $F_n = q_1 \cdot \dots \cdot q_k$, where $q_1 < \dots < q_k$ are Mersenne primes. Let

$q_i = 2^{p_i} - 1$, where $p_i \geq 2$ is prime. In particular, $q_i \equiv 3 \pmod 4$. Reducing equation (9) modulo q_i, it follows that

$$L_n^2 = (-1)^n \cdot 4 \pmod{q_i}. \tag{11}$$

Since $q_i \equiv 3 \pmod 4$, it follows that $\left(\frac{-1}{q_i}\right) = -1$. From congruence 11, it follows that $2 \mid n$. Let $n = 2n_1$. Since $F_n = F_{2n_1} = F_{n_1}L_{n_1}$ and since F_n is a square free product of Mersenne primes, it follows that F_{n_1} is a square free product of Mersenne primes as well. In particular, $\sigma(F_{n_1}) = 2^{m_1}$. Inductively, it follows easily that n is a power of 2. Let $n = 2^t$, where $t \geq 4$. Then, $n_1 = 2^{t-1}$. Moreover, since $L_{n_1} \mid F_{n_1}L_{n_1} = F_n$, it follows that L_{n_1} is a square free product of Mersenne primes as well. Write

$$L_{n_1} = q_1' \cdot \dots \cdot q_l', \tag{12}$$

where $q_i' < \dots < q_l'$. Let $q_i' = 2^{p_i'} - 1$ for some prime number p_i'. The sequence $(L_n)_{n \geq 0}$ is periodic modulo 3 with period 8. Moreover, analyzing L_s for $s = 0, 1, \dots, 7$, one concludes that $3 \mid L_s$ only for $s = 2, 6$. Hence, $3 \mid L_s$ if and only if $s \equiv 2 \pmod 4$. Since $t \geq 4$, it follows that $8 \mid 2^{t-1} = n_1$. Hence, $3 \nmid L_n$ and $3 \nmid L_{n_1/2}$. In particular, $p_1' > 2$. We conclude that all p_i' are odd and $q_i' = 2^{p_i'} - 1$ $\equiv 2 - 1 \equiv 1 \pmod 3$. From equation (12), it follows that $L_{n_1} \equiv 1 \pmod 3$. Reducing relation $L_{n_1} = L_{n_1/2}^2 - 2$ modulo 3, it follows that $1 \equiv 1 - 2 \equiv -1 \pmod 3$, which is a contradiction.

(4) We first show that equation (4) has no solutions for which $n > 1$ is odd. Indeed, assume that $\sigma(L_n) = 2^m$ for some odd integer n. Let $p \mid n$ be a prime. By Lemma 2(2), we conclude that $L_p \mid L_n$. Since $\sigma(L_n)$ is a power of 2, it follows that L_n is a square free product of Mersenne primes. Since L_p is a divisor of L_n, it follows that L_p is a square free product of Mersenne primes as well. Write $L_p = q_1 \cdot \dots \cdot q_k$, where $q_1 < \dots < q_k$ are prime numbers such that $q_i = 2^{p_i} - 1$ for some prime $p_i \geq 2$. We show that $p_1 > 2$. Indeed, assume that $p_1 = 2$. In this case, $q_1 = 3$. It follows that $3 \mid L_p$. However, from the proof of (3), we know that $3 \mid L_s$ if and only if $s \equiv 2$ (mod 4). This shows that $p_1 \geq 3$.

Notice that $L_p \equiv \pm 1 \pmod{2^{p_1}}$. It follows that $L_p^2 - 1 \equiv 0 \pmod{2^{p_1+1}}$. Since p is odd, it follows, by Lemma 1(4), that

$$L_p^2 - 5F_p^2 = -4 \tag{13}$$

or $L_p^2 - 1 = 5(F_p^2 - 1)$. It follows that $F_p^2 - 1 \equiv 0 \pmod{2^{p_1+1}}$. Hence, $F_p \equiv \pm 1 \pmod{2^{p_1}}$. From Lemma 3(3), we conclude that $p \equiv \pm 1 \pmod{2^{p_1-1}3}$. In particular,

$$p \geq 2^{p_1-1}3 - 1. \tag{14}$$

On the other hand, reducing equation (13) modulo q_1, we conclude that $5F_p^2 \equiv 4 \pmod{q_1}$, therefore $\left(\frac{5}{q_1}\right) = 1$. By; Lemma 2(1), it follows that $q_1 \mid F_{q_1-1}$. Since $q_1 \mid L_p$ and $F_{2p} = F_pL_p$, it follows that $q_1 \mid F_{2p}$. Hence, $q_1 \mid (F_{2p}, F_{q_1-1}) = F_{(2p, q_1-1)}$. Since $F_2 = 1$, we conclude that $p \mid q_1 = 1 = 2(2^{p_1-1} - 1)$. In particular,

$$p \leq 2^{p_1-1} - 1. \tag{15}$$

From inequalities (14) and (15), it follows that $2^{p_1-1}3 - 1 \leq 2^{p_1-1} - 1$, which is a contradiction.

Assume now that $n > 4$ is even. Write $n = 2^t n_1$, where n_1 is odd. Let

$$L_n = q_1 \cdot \dots \cdot q_k, \tag{16}$$

where $q_1 < \cdots < q_k$ are prime numbers of the Mersenne type. Let $q_i = 2^{p_i} - 1$. Clearly, $q_i \equiv 3 \pmod 4$ for all $i = 1, \ldots, k$. Reducing the equation $L_n^2 - 5F_n^2 = 4$ modulo q_i, we obtain that $-5F_n^2 = 4 \pmod{q_i}$. Since $\left(\frac{-1}{q_i}\right) = -1$, it follows that $\left(\frac{5}{q_i}\right) = -1$. From Lemma 2(1), we conclude that $q_i \mid F_{q_i+1} = F_{2p_i}$. We now show that $t \le p_1 - 1$. Indeed, assume that this is not the case. Since $t \ge p_1$, it follows that $2^{p_i} \mid 2^t n_1 = n$. Hence, $q_1 \mid F_{2p_1} \mid F_n$. Since $q_1 \mid L_n$, it follows, by Lemma 1(4), that $q_1 \mid 4$, which is a contradiction. So, $t \le p_1 - 1$. We now show that $n_1 = 1$. Indeed, since $t + 1 \le p_1 \le p_i$, $q_i \mid L_n \mid F_{2n}$, and $q_i \mid F_{2p_i}$, it follows, by Lemma 2(2), that $q_i \mid (F_{2n}, F_{2p_i}) = F_{(2n, 2^{p_i})} = F_{2^{t+1}}$. Hence, $q_i \mid F_{2^{t+1}} = F_{2^t} L_{2^t}$. We show that $n_1 = 1$. Indeed, since $t + 1 \le p_1 \le p_i$, $q_i \mid L_n \mid F_{2n}$, and $q_i \mid F_{2p_i}$, it follows, by Lemma 2(2), that $q_i \mid (F_{2n}, F_{2p_i}) = F_{(2n, 2^{p_i})} = F_{2^{t+1}}$. Hence, $q_i \mid F_{2^{t+1}} = F_{2^t} L_{2^t}$. We show that $q_i \mid L_{2^t}$. Indeed, for if not, then $q_i \mid F_{2^t}$. Since $2^t \mid n$, it follows that $q_i \mid F_{2^t} \mid F_n$. Since $q_i \mid L_n$, it follows, by Lemma 1(4), that $q_i^2 \mid 4$, which is a contradiction. In conclusion, $q_i \mid L_{2^t}$ for all $i = 1, \ldots, k$. Since q_i are distinct primes, it follows that

$$L_n = q_1 \cdot \cdots \cdot q_k \mid L_{2^t}.$$

In particular, $L_{2^t} \ge L_n = L_{2^t n_1}$. This shows that $n_1 = 1$. Hence, $n = 2^t$.

Since $n > 4$; it follows that $t \ge 3$. It is apparent that $q_1 \ne 3$, since, as previously noted, $3 \mid L_s$ if and only is $s \equiv 2 \pmod 4$, whereas $n = 2^t \equiv 0 \pmod 4$. Hence, $p_i \ge 3$ for all $i = 1, \ldots, k$. Moreover, since $q_i = 2^{p_i} - 1$ are quadratic nonresidues modulo 5, it follows easily that $p_i \equiv 3 \pmod 4$. In particular, if $k \ge 2$, then $p_2 \ge p_1 + 4$.

Now since $t \ge 3$, it follows, by Lemma 4, that

$$L_{2^t} \equiv 2^{t+1} a - 1 \pmod{2^{t+4}}, \tag{17}$$

where $a \in \{3, 5\}$. On the other hand, from formula (16) and the fact that $p_2 \ge p_1 + 4$ whenever $k \ge 2$, it follows that

$$L_{2^t} = \prod_{i=1}^{k} (2^{p_i} - 1) \equiv (-1)^k \cdot (-2^{p_1} + 1) \equiv 2^{p_1} b \pm 1 \pmod{2^{p_1+4}}. \tag{18}$$

where $b \in \{1, 7\}$. One can notice easily that congruences (17) and (18) cannot hold simultaneously for any $t \le p_1 - 1$. This argument takes care of the situation $k \ge 2$. The case $k = 1$ follows from Lemma 3 and the fact that $t \le p_1 - 1$ by noticing that

$$2^{p_1} - 1 = L_{2^t} \equiv 2^{t+1} \cdot 3 - 1 \pmod{2^{t+4}}$$

implies $2^{p_1 - t - 1} \equiv 3 \pmod{2^3}$, which is impossible.

The above arguments show that equation (4) has no even solutions $n > 4$. Hence, the only solutions are the announced ones. □

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