RANDOM COMBINATIONS WITH BOUNDED DIFFERENCES AND COSPAN

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(Submitted May 1998-Final Revision November 1998)

1. INTRODUCTION

Let

$$1 \le x_1 < x_2 < \cdots < x_r \le n \tag{1.1}$$

be an r combination from n. Moser and Abramson [17] used the terms *differences* for the quantities $d_j = x_{j+1} - x_j$, $j = 1, 2, \ldots, r-1$, *span* for $d = x_r - x_1$, and *cospan* for $n - d$. It is clear that $d_j - 1$ is, for $j = 1, 2, \ldots, r-1$, the number of integers, from $\{1, 2, \ldots, n\}$, which lie "between" x_j and x_{j+1}. Similarly, $(n-d)-1$ is the number of integers from x_r clockwise to x_1.

To the r-combination (1.1) there corresponds a unique *place-indicator vector* $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ defined by

$$\varepsilon_i = \begin{cases} 1 & \text{if } i = x_1, \ldots, x_r, \\ 0 & \text{otherwise.} \end{cases}$$

Assume that $\varepsilon_i, i = 1, 2, \ldots, n$ are generated by a random process in which the outcome of the t^{th} trial depends on the outcomes of the previous trials in a first-order Markovian fashion. Moreover, let

$$p_j(1) = P(\varepsilon_1 = j), \quad j = 0, 1, \tag{1.2}$$

denote the initial probabilities and

$$p_{ij}(t) = P(\varepsilon_t = j \mid \varepsilon_{t-1} = i), \quad i, j \in \{0, 1\}, \ t = 2, \ldots, n, \tag{1.3}$$

denote the first-order transition probabilities of the process. By the term *random combination* we shall refer to the combination associated with $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, i.e., integer j $(1 \le j \le n)$ will be selected if and only if $\varepsilon_j = 1$.

Let $A_n(k, k'; l, l')$ denote the event that the random combination associated with the sequence $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ satisfies the conditions

$$k \le d_j \le k', \quad j = 1, 2, \ldots, r-1 \text{ and } l \le n - d \le l',$$

where k, k', l, l' are pre-specified integers $(1 \le k \le k', \ 1 \le l \le l')$ and r is the number of nonzero entities in $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$.

The probability of the event $A_n(k, k'; l, l')$ is, in certain special cases, very closely related to some problems of interest in combinatorial analysis, statistical theory of runs and reliability theory. Thus, for the symmetric i.i.d. case, *viz.*,

$$p_j(1) = p_{ij}(t) = \frac{1}{2}, \quad i, j \in \{0, 1\}, \ t = 2, \ldots, n,$$

the quantity

$$2^n P[A_n(k, k'; l, l')]$$

enumerates the (nonrandom) combinations from n whose differences and cospan satisfy the conditions $k \le d_1, d_2, \ldots, d_{r-1} \le k'$ and $l \le n - d \le l'$. More specifically, if $C_{n,r}(k, k'; l, l')$ is the number of r-combinations (1.1) with $k \le d_j \le k'$, $j = 1, 2, \ldots, r-1$, and $l \le n - d \le l'$ (see [17]), then

$$2^n P[A_n(k, k'; l, l')] = \sum_{r=0}^{n} C_{n,r}(k, k'; l, l').$$

Now let us consider the first-order Markov dependence model and assume that l' is sufficiently large so that $n - d \le l'$ poses no restriction on the cospan. Then, $P[A_n(1, k'; 1, l')]$ turns out to be the probability that the *longest run*[1] of 0's, in n Markov dependent trials $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ does not exceed $k' - 1$; for related problems, see [1], [2], [8], [16], and the references therein. Similarly, $P[A_n(1, k'; 1, k')]$ turns out to be the corresponding probability for the *longest circular run*, i.e., when the trials $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are arranged in a circular fashion so that ε_1 becomes adjacent to ε_n (see [13], [21]). In the i.i.d. case,

$$p_0(1) = p_{i0}(t) = q, \ p_1(1) = p_{i1}(t) = p, \ i \in \{0, 1\}, \ t = 2, \ldots, n, \tag{1.4}$$

the probabilities mentioned above are closely related to the reliability of a linear/circular consecutive-k'-out-of-n:F reliability system with component failure probabilities q; for a review on this topic, one may refer to Chao et al. [3].

Finally, let us assume that both k' and l' are sufficiently large so that $d_j \le k'$ and $n - d \le l'$ are practically no restrictions. Then, the occurrence of the event $A_n(k, k'; 1, l')$ implies that the length of the *shortest run* of 0's in the sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ is at least $k - 1$ or, equivalently, there exists no pair of 1's separated by $k - 2$ or less 0's. A related waiting time problem was recently studied by Koutras [10]. In the i.i.d. case, $P[A_n(k, k'; 1, l')]$ coincides with the reliability of a 2-within-consecutive-k-out-of-n:F system with component failure probabilities p (see [3], [19], and [20]); it is also related to sliding window probabilities [18] and scan statistics [5].

The purpose of the present paper is to conduct a detailed study of the probability of the event $A_n(k, k'; k, l')$ when k' and l' are sufficiently large. In this case, we shall use the notation $A_n(k)$ for the event. It is clear that the occurrence of $A_n(k)$ implies that the length of the *shortest circular run* of 0's in the sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ is at least $k - 1$.

In Section 2, we introduce the necessary notations and develop formulas for the evaluation of $P[A_n(k)]$ in the general case of Markov dependent trials. In Section 3, we restrict ourselves to a homogeneous Markov-dependence model and derive the generating function of the sequence $\{P[A_n(k)]\}_{n \ge k}$. In addition, a set of recurrence relations is established which offers a computationally efficient scheme for the calculation of $P[A_n(k)]$. In Section 4, we focus our attention on the circular 2-within-consecutive-k-out-of-n:F system. Finally, in Section 5, we express $P[A_n(k)]$ in terms of appropriate generalizations of Lucas polynomials and numbers.

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[1] Here, by the term "run of 0's" we mean a string of consecutive 0's preceded and followed by 1's.

2. GENERAL MODEL

Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be a finite sequence of Markov dependent random variables with initial distribution (1.2) and first-order transition probabilities (1.3). By $R_n(k)$, or simply R_n, if no confusion is likely to arise, we shall denote the probability that the differences and cospan of the associated random r-combinations ($r = 0, 1, \ldots, n$) are at least k, i.e., $d_j \geq k$, $j = 1, 2, \ldots, r-1$, and $n - d \geq k$. By convention, for $r = 0$, we assume that there is only one r-combination satisfying the aforementioned conditions (the one that corresponds to the place indicator $(0, 0, \ldots, 0)$). On the other hand, for $r = 1$, we treat all 1-combinations (i.e., the ones associated with the place indicators $(1, 0, \ldots, 0)$, $(0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)$) as valid choices for $A_n(k)$. It is worth stressing that this setup is slightly different from the one used by Moser and Abramson [17], who assumed that the 1-combinations are acceptable choices for $n \geq k$ and nonacceptable for $n < k$.

Employing the notation of the last section, we may write

$$R_n = R_n(k) = P[A_n(k)] = P[A_n(k, l; k, l')]$$

with l, l' being sufficiently large.

In order to evaluate R_n, we will employ a Markov chain approach similar to the one used by Koutras [10] for the study of several reliability systems; see also [4] and [12] for additional applications of the same method to success runs enumeration problems.

Observe first that, for $n < k$, we have

$$\begin{aligned} &R_n = 1 \text{ for } n = 0, 1, \\ &R_n = 1 - p_1(1)p_{11}(2) \text{ for } n = 2, \\ &R_n = p_0(1)\sum_{t=2}^{n} p_{00}(t) + p_1(1)p_{10}(2)\sum_{t=3}^{n} p_{00}(t) + p_0(1)\sum_{i=2}^{n-1}\left[\prod_{t=2, t\neq i, i+1}^{n} p_{00}(t)\right] \\ &\quad \times p_{01}(i)p_{10}(i+1) + p_0(1)\left[\prod_{t=2}^{n-1} p_{00}(t)\right]p_{01}(n) \text{ for } 3 \leq n < k. \end{aligned} \tag{2.1}$$

The first two expressions are obvious. For the third one, it is enough to observe that the occurrence of $A_n(k)$, $3 \leq n < k$, secures that at most one trial among $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ resulted in 1. The required formula is then easily gained by conditioning on the position where 1 should be placed.

Next, assume that $n \geq k$. By introducing the events

$$\begin{aligned} &B_0 : \varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_k = 0, \\ &B_i : \varepsilon_{k-i+1} = 1 \quad \text{and} \quad \varepsilon_j = 0, \ j \in \{1, 2, \ldots, k\} \setminus \{k - i + 1\} \end{aligned}$$

for $i = 1, 2, \ldots, k$, we may write

$$R_n = \sum_{i=0}^{k} \beta_i P[A_n(k) \mid B_i], \tag{2.2}$$

where $\beta_i = P(B_i)$ are given by

$$\beta_i = \begin{cases} p_0(1)\prod_{t=2}^{k} p_{00}(t) & \text{for } i = 0, \\ p_0(1)\left[\prod_{t=2}^{k-1} p_{00}(t)\right]p_{01}(k) & \text{for } i = 1, \\ p_0(1)\left[\prod_{t=2, t\neq k-i+1, t\neq k-i+2}^{k} p_{00}(t)\right] & \\ \quad \times p_{01}(k-i+1)p_{10}(k-i+2) & \text{for } 2 \leq i \leq k-1, \\ p_1(1)p_{10}(2)\prod_{t=3}^{k} p_{00}(t) & \text{for } i = k. \end{cases} \tag{2.3}$$

For the evaluation of the conditional probabilities $P[A_n(k) \mid B_i]$, $i = 0, 1, \ldots, k$, we introduce a Markov chain $\{Y_t,\ t = 1, 2, \ldots\}$ defined on the finite state space $\Omega = \{1, 2, \ldots, k+2\}$ as follows:

$Y_t = 1$ if $\varepsilon_i = 0$ for $\max(1, t-k+1) \le i \le t$;

$Y_t = j$ if $\varepsilon_{t-j+2} = 1$ and $\varepsilon_i = 0$ for $i \ne t-j+2$, $\max(1, t-k+1) \le i \le t$ $(2 \le j \le k+1)$;

$Y_t = k+2$ if there exist indices t_1, t_2 with $\max(1, t-k+1) \le t_1 \ne t_2 \le t$ such that $\varepsilon_{t_1} = \varepsilon_{t_2} = 1$.

(Note that states $j, 2 \le j \le k+1$, are only reachable after time $t \ge j-1$.) Let us denote by Λ_t the transition probability of the aforementioned Markov chain, i.e.,

$$\Lambda_t = (p(Y_t = j \mid Y_{t-1} = i))_{(k+2)\times(k+2)}.$$

From the description of the states, we may immediately verify that Λ_t is given by

$$\Lambda_t = \begin{bmatrix} p_{00}(t) & p_{01}(t) & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & p_{10}(t) & 0 & \cdots & 0 & 0 & 0 & p_{11}(t) \\ 0 & 0 & 0 & p_{00}(t) & \cdots & 0 & 0 & 0 & p_{01}(t) \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & p_{01}(t) \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & p_{00}(t) & 0 & p_{01}(t) \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & p_{00}(t) & p_{01}(t) \\ p_{00}(t) & p_{01}(t) & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The conditional probabilities $P[A(k) \mid B_i]$, $i = 0, 1, \ldots, k$, can now be expressed by means of higher order transition probability matrices or, equivalently, products of Λ_t's. Thus, denoting by $\mathbf{e}_j$ the j^{th} unit (row) vector of the space $\mathbf{R}^{k+2}$, $\mathbf{u} = (1, 1, \ldots, 1, 0) = \sum_{j=1}^{k+1} \mathbf{e}_j$, and $\mathbf{u}_i = \mathbf{u} - \sum_{j=2}^{i} \mathbf{e}_j$, $i = 2, \ldots, k+1$, we obtain

$$\begin{aligned} P(A_n(k) \mid B_i) &= P(Y_n \ne k+2 \mid Y_k = i+1) = \mathbf{e}_{i+1}\left(\prod_{t=k+1}^{n} \Lambda_t\right)\mathbf{u}' && \text{for } i = 0, 1, \\ P(A_n(k) + B_i) &= P(Y_n \in \{1, i+1, \ldots, k+1\} \mid Y_k = i+1) = \mathbf{e}_{i+1}\left(\prod_{t=k+1}^{n} \Lambda_t\right)\mathbf{u}_i' && \text{for } 2 \le i \le k. \end{aligned} \tag{2.4}$$

A combined use of formulas (2.2), (2.3), and (2.4) offer a compact computational scheme for the evaluation of R_n.

It is worth mentioning that, on introducing the convention $\mathbf{u}_0 = \mathbf{u}_1 = \mathbf{u}$, we may write a unified formula for R_n,

$$R_n = \sum_{i=0}^{k} \beta_i \mathbf{e}_{i+1}\left(\prod_{t=k+1}^{n} \Lambda_t\right)\mathbf{u}_i'. \tag{2.5}$$

Note also that, for $n \le 2k-1$, one does not have to use (2.5) for the evaluation of R_n, since the third part of (2.1) is valid for $k \le n \le 2k-1$ as well.

In closing, we mention that the technique employed here for the study of the event $A_n(k) = A_n(k, k'; k, l')$ can be modified effortlessly to capture the probability of the more general event $A_n(k, k'; l, l')$. The consideration of the special case was for typographical convenience only.

3. HOMOGENEOUS MARKOV-DEPENDENCE MODEL

In this section, we study in some detail the special case in which $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ form a homogeneous first-order Markov sequence, i.e., $p_{ij}(t)$ are independent of t. Using the notation $p_j - p_j(1)$, $p_{ij} = p_{ij}(t)$, $t = 2, 3, \ldots, n$, for $i, j \in \{0, 1\}$, we may write (2.1) as follows:

$$\begin{aligned} &R_n = 1 \text{ if } n = 0, 1; \\ &R_n = 1 - p_1 p_{11} \text{ if } n = 2; \\ &R_n = (p_0 + p_1 p_{10}) p_{00}^{n-2} + (n-2) p_0 p_{01} p_{10} p_{00}^{n-3} \text{ if } 3 \le n < k. \end{aligned} \tag{3.1}$$

Since $\Lambda_t = \Lambda$ for all $t = 2, 3, \ldots, n$, we have $P(A_n(k) \mid B_i) = \mathbf{e}_{i+1} \Lambda^{n-k} \mathbf{u}_i'$, $n \ge k$, and (2.5) leads to the expression

$$R_n = \sum_{i=0}^{k} \beta_i \mathbf{e}_{i+1} \Lambda^{n-k} \mathbf{u}_i', \quad n \ge k, \tag{3.2}$$

with the β_i's given by

$$\beta_i = \begin{cases} p_0 p_{00}^{k-1} & \text{if } i = 0, \\ p_0 p_{00}^{k-2} p_{01} & \text{if } i = 1, \\ p_0 p_{00}^{k-3} p_{01} p_{10} & \text{if } 2 \le i \le k-1, \\ p_1 p_{10} p_{00}^{k-2} & \text{if } i = k. \end{cases}$$

From (2), we can easily obtain an explicit expression for the generating function

$$G_k(x) = \sum_{n=k}^{\infty} R_n z^n. \tag{3.3}$$

Now, using (3.2) in (3.3), interchanging the order of summation and then substituting the resulting geometric (matrix) series, we obtain the final expression for the generating function as

$$G_k(z) = z^k \sum_{i=0}^{k} \beta_i \mathbf{e}_{i+1} (I - \Lambda z)^{-1} \mathbf{u}_i'.$$

After somewhat lengthy but straightforward algebraic calculations on the matrix $I - \Lambda z$, we get

$$G_k(z) = \frac{\{(p_0 + p_1 p_{10}) p_{00}^{k-2} + (k-2) p_0 p_{01} p_{10} p_{00}^{k-3}\} z^k + p_0 p_{01} p_{10} \sum_{n=k+1}^{2k-1} p_{00}^{n-3} z^n}{1 - p_{00} z - p_{01} p_{10} p_{00}^{k-2} z^k}. \tag{3.4}$$

From (3.4), we can easily derive a recursive scheme for the evaluation of R_n. Multiplying both sides of (3.4) by the denominator, using (3.3) on the left-hand side, and equating the coefficients of z^n on both sides, we obtain

$$\begin{aligned} &R_k = p_0 p_{00}^{k-1} + p_{00}^{k-3} [p_0 p_{01} p_{00} + (k-2) p_0 p_{01} p_{10} + p_1 p_{10} p_{00}], \\ &R_n = p_{00} R_{n-1} + p_0 p_{01} p_{10} p_{00}^{n-3} \quad \text{for } k+1 \le n \le 2k-1, \\ &R_n = p_{00} R_{n-1} + p_{01} p_{10} p_{00}^{k-2} R_{n-k} \quad \text{for } n \ge 2k. \end{aligned} \tag{3.5}$$

Evaluation of R_n through (3.5) is preferable, instead of through (3.2), due to simplicity (no matrix multiplications are necessary) as well as accuracy (less round-off errors). It may also be noted that, for $2 \le n \le 2k-1$, one can use the exact formula

$$R_n = p_0 p_{00}^{n-1} + p_{00}^{n-3}[p_0 p_{01} p_{00} + (n-2) p_0 p_{01} p_{10} + p_1 p_{10} p_{00}]$$

and employ the recursive scheme in (3.5) only for $n \geq 2k$.

4. CIRCULAR 2-WITHIN-CONSECUTIVE-k-OUT-OF-n:F SYSTEM

An r-within-consecutive-k-out-of-n:F system fails if and only if there exist k consecutive components which include among them at least r failed ones. Applications of such structures have been well documented in the literature, which include applications to telecommunications, design of integrated circuits, quality control, and sliding window detectors (see, e.g., [5], [6], [9], [18], [19], [20], and [24]).

Even for the case of linearly arranged components, the evaluation of system's reliability is a very difficult task, and is mainly performed through approximating formulas. A Markov chain approach for this problem can be found in [10] along with recurrence relations for the special case $r = 2$.

The results of Section 2 can be used for the reliability evaluation of a *circular* 2-within-consecutive-k-out-of-n:F system. Let us assume that the n components of the system work independently and denote by $p = 1 - q$ their failure probabilities (i.i.d. model). It is clear, from the definition of the event $A_n(k)$, that in the special case $p_1 = p_{01} = p_{11} = p$, $p_0 = p_{00} = p_{10} = q$, R_n is exactly the same as the reliability of a circular 2-within-consecutive-k-out-of-n:F system. Recurrence relations in (3.5) reduce in this case to

$$\begin{aligned} R_n &= qR_{n-1} + pq^{n-1} && \text{if } k+1 \leq n < 2k, \\ R_n &= qR_{n-1} + pq^{k-1}R_{n-k} && \text{if } n \geq 2k, \end{aligned} \tag{4.1}$$

with initial conditions $R_0 = R_1 = 1$, $R_n = q^n + npq^{n-1}$, $2 \leq n \leq k$. Note again that instead of the first recurrence relation in (4.1), one could use the exact formula $R_n = q^n + npq^{n-1}$ for all $2 \leq n < 2k$. Another interesting observation to be made here is that, for $n \geq 2k$, the reliability of both linear and circular 2-within-consecutive-k-out-of-n:F systems satisfy exactly the same recurrence relation. This is not surprising since, when the system becomes sufficiently large ($n \geq 2k$), the transition from R_{n-1} to R_n is not affected by the topological arrangement (adjacent or not) of components 1, n.

Recurrence relations in (4.1) can be used in conjunction with the obvious inequality $R_{n-k} \geq R_{n-1}$, $k \geq 1$, in order to establish some simple lower bounds for R_n. Thus, for $n \geq 2k$, we have $R_n \geq (q + pq^{k-1})R_{n-1}$; repeated application of this argument yields $R_n \geq (q + pq^{k-1})^{n-2k+1} R_{2k-1}$, $n \geq 2k$, which, when used with the result that $R_{2k-1} = q^{2k-1} + (2k-1)pq^{2k-2}$, gives a lower bound for R_n as

$$R_n \geq (q + pq^{k-1})^{n-2k+1}\{q^{2k-1} + (2k-1)pq^{2k-2}\}, \quad n \geq 2k. \tag{4.2}$$

This bound is very useful when addressing the problem of specifying the (maximum) size n of the system warrantying a prespecified level a $(0 < a < 1)$ of reliability. In view of (4.2), the condition $R_n \geq a$ will be met if we force the right-hand side of (4.2) to exceed a; upon then solving for n, we obtain

$$n \leq (2k-1) + \left(\log \frac{a}{q^{2k-1} + (2k-1)pq^{2k-2}}\right)\{\log(q + pq^{k-1})\}^{-1}.$$

It needs to be mentioned here that if $R_{2k-1} < a$ then appropriate values of n should be sought with the aid of the exact formula $R_n = q^n + npq^{n-1}$, $2 \le n < 2k$.

Papastavridis and Koutras [19] derived upper and lower bounds for both linear and circular r-within-consecutive-k-out-of-n:F systems. In the special case in which $r = 2$, their lower bound for the circular system becomes

$$R_n \ge (q + pq^{k-1})^{n-1}(q^k + kpq^{k-1}), \quad n \ge 2. \tag{4.3}$$

These authors also established a Weibull limit theorem for system's lifetime under quite general assumptions on components' lifetime distributions. A simple adjustment to their proof yields the following asymptotic result: If p depends on n in such a way that $\lim_{n\to\infty} np^2 = \lambda > 0$, then

$$\lim_{n\to\infty} R_n = e^{-(k-1)\lambda}. \tag{4.4}$$

Simple algebraic calculations on the lower bounds in (4.2) and (4.3) reveal that, under the condition $\lim_{n\to\infty} np^2 = \lambda$, they converge to the limiting value given in (4.4).

In Table 1, a numerical comparison of the lower bounds in (4.2) and (4.3) is performed for selected values of n, k, and p. The exact value of R_n and the limiting value $e^{-(k-1)np^2}$ are also provided for comparison purposes.

5. LUCAS POLYNOMIALS AND NUMBERS

Let $\{L_n^{(k)}(x)\}_{n\ge 0}$ be the sequence of polynomials defined recursively as follows:

$$\begin{aligned} L_n^{(k)}(x) &= 1 + nx && \text{for } 0 \le n < 2k, \\ L_n^{(k)}(x) &= L_{n-1}^{(k)}(x) + xL_{n-k}^{(k)}(x) && \text{for } n \ge 2k. \end{aligned} \tag{5.1}$$

It may be seen readily that the degree of $L_n^{(k)}(x)$ for $0 \le n < 2k$ is 1; moreover, if $sk \le n < (s+1)k$, $s = 2, 3, \ldots$, the degree of $L_n^{(k)}(x)$ is s.

Next, let us denote by $L_n^{(k)}$ the integers $L_n^{(k)}(1)$, $n \ge 0$. Then $\{L_n^{(k)}\}_{n\ge 0}$ will satisfy the recurrence relation

$$L_n^{(k)} = L_{n-1}^{(k)} + L_{n-k}^{(k)}, \quad n \ge 2k, \tag{5.2}$$

with initial conditions

$$L_n^{(k)} = n + 1, \quad 0 \le n < 2k. \tag{5.3}$$

It is clear that, for $k = 2$, the corresponding numbers $L_n^{(2)}$, $n \ge 2$, coincide with the well-known Lucas numbers L_n. Hence, an appropriate name for the numbers $L_n^{(k)}$ seems to be *k-step Lucas numbers*. Likewise, $L_n^{(k)}(x)$ may aptly be called *k-step Lucas polynomials*.

It is worth noting that the recurrence relation in (5.2), under different initial conditions, gives rise to analogous generalizations of Fibonacci numbers. However, they have been studied in the literature under many different names. For example, Mohanty [15] termed them *generalized Fibonacci numbers* (see also Roselle [23] and Moser and Abramson [17]) and proved the existence of minimal and maximal representations of positive integers as sums of such numbers; Hasunuma and Shibata [7] used the name k^{th} *interspaced Fibonacci numbers*, while Koutras [11] employed the term *k-step Fibonacci numbers*.

Hasunuma and Shibata [7] defined a Lucas number analogue as well, by considering the sequence $L_n^{\langle k\rangle}$ satisfying the recurrence $L_n^{\langle k\rangle} = L_{n-1}^{\langle k\rangle} + L_{n-k}^{\langle k\rangle}$, $n \ge 2$, with initial conditions $L_0^{\langle k\rangle} = k$,

$L_1^{\langle k\rangle} = 1$, and $L_n^{\langle k\rangle} = 0$ for $n < 0$. The numbers $L_n^{\langle k\rangle}$, names as k^{th} *interspaced Lucas numbers* by them, arise in a very natural way in an interesting graph theoretic problem. Specifically, in [7], Hasunuma and Shibata proved that the number of labeled graphs which are k-placeable by a given permutation is a product of interspaced k^{th} Lucas numbers. It can be shown that the k-step Lucas numbers $L_n^{(k)}$ (defined above) and the k^{th} interspaced Lucas numbers $L_n^{\langle k\rangle}$ (just defined) coincide for $n \geq k$, that is, $L_n^{(k)} = L_n^{\langle k\rangle}$ for all $n = k, k+1, \ldots$.

The generating function of the sequence $\{L_n^{(k)}(x)\}_{n\geq k}$ given by

$$G(z; x) = \sum_{n=k}^{\infty} L_n^{(k)}(x) z^n$$

is readily determined from (5.1) to be

$$G(z; x) = \frac{z^k[1 + kx + xz(1 + z + \cdots + z^{k-2})]}{1 - z - xz^k}.$$

Comparing $G(z; x)$ to the generating function of R_n for the i.i.d. case, we obtain the relationship

$$\sum_{n=k}^{\infty} R_n z^n = \frac{(qz)^k[1 + k_q^p + pz(1 + qz + \cdots + (qz)^{k-2})]}{1 - qz - pq^{k-1}z^k} = G\left(qz; \frac{p}{q}\right).$$

Hence,

$$\sum_{n=k}^{\infty} R_n z^n = \sum_{n=k}^{\infty} L_n^{(k)}\left(\frac{q}{p}\right)(qz)^n$$

and the reliability R_n of a 2-within-consecutive-k-out-of-n:F system can be expressed in terms of k-step Lucas polynomials as follows:

$$R_n = q^n L_n^{(k)}\left(\frac{p}{q}\right), \quad n \geq k.$$

This formula yields an interesting combinatorial interpretation for the k-step numbers $L_n^{(k)}$. More specifically, considering the symmetric case $p = q = 1/2$, we note from the above relation that $L_n^{(k)} = 2^n R_n = 2^n P[A_n(k)]$, which simply proves that $L_n^{(k)}$ is the total number of "*circular*" combinations whose differences and cospan are at least k. Moser and Abramson [17] arrived at the same conclusion by first computing the number $C_r(k, k'; k, l')$ of r-combinations with differences and cospan at least k (k' and l' are assumed sufficiently large so that the differences and cospan are practically unbounded from above) and then noting that

$$L_n^{(k)} = \sum_{r=0}^{n} C_r(k, k'; k, l') \tag{5.4}$$

satisfies (5.2) and (5.3). It should be stressed that, due to the different conventions used here for the 1-combinations (c.f. first paragraph of Section 2), our numbers $L_n^{(k)}$ coincide with the ones appearing in [17] only for $n \geq k$. Analogous results can be found in [22] and [25].

In closing, we note that the generating function of the k-step Lucas numbers $L_n^{(k)}$, $n \geq k$, is given by

$$\sum_{n=k}^{\infty} L_n^{(k)} z^n = G(z; 1) = \frac{z^k[1 + k + z(1 + z + \cdots + z^{k-2})]}{1 - z - z^k}.$$

Expanding the right-hand side in a power series around 0, we easily get an explicit expression for $L_n^{(k)}$ as

$$L_n^{(k)} = 1 + \sum_{r \geq 1} \frac{n}{r} \binom{n - r(k-1) - 1}{r-1}. \tag{5.5}$$

This formula was also derived by Moser and Abramson [17] using direct combinatorial arguments. In fact, they first proved that

$$C_r(k, k'; k, l') = \frac{n}{r} \binom{n - r(k-1) - 1}{r-1}, \quad r \geq 1,$$

and then employed (5.4) in order to derive the explicit expression in (5.5).

TABLE 1. Comparison of Lower Bounds of Section 4 and Exact and Limiting Values of R_n

n	k	p	*Exact Value* R_n	$e^{-(k-1)np^2}$	*Lower Bound* (2)	*Lower Bound* (3)
4	2	0.01	0.9996	0.9996	0.9996	0.9996
		0.05	0.9905	0.9900	0.9903	0.9900
		0.10	0.9639	0.9608	0.9623	0.9606
		0.20	0.8704	0.8521	0.8602	0.8493
		0.50	0.4375	0.3679	0.3750	0.3164
		0.80	0.0784	0.0773	0.0374	0.0168
		0.90	0.0199	0.0392	0.0053	0.0013
		0.95	0.0050	0.0271	0.0007	0.0001
		0.99	0.0002	0.0198	0.0000	0.0000

n	k	p	*Exact Value* R_n	$e^{-(k-1)np^2}$	*Lower Bound* (2)	*Lower Bound* (3)
6	2	0.01	0.9994	0.9994	0.9994	0.9994
		0.05	0.9858	0.9851	0.9853	0.9851
		0.10	0.9462	0.9418	0.9431	0.9415
		0.20	0.8110	0.7866	0.7927	0.7828
		0.50	0.2813	0.2231	0.2109	0.1780
		0.80	0.0190	0.0215	0.0049	0.0022
		0.90	0.0022	0.0078	0.0002	0.0000
		0.95	0.0003	0.0044	0.0000	0.0000
		0.99	0.0000	0.0028	0.0000	0.0000
	3	0.01	0.9988	0.9988	0.9988	0.9987
		0.05	0.9733	0.9704	0.9726	0.9688
		0.10	0.9054	0.8869	0.9011	0.8831
		0.20	0.7045	0.6188	0.6842	0.6167
		0.50	0.1563	0.0498	0.1172	0.0477
		0.80	0.0047	0.0005	0.0016	0.0001
		0.90	0.0003	0.0001	0.0001	0.0000
		0.95	0.0000	0.0000	0.0000	0.0000

TABLE 1 (continued)

n	k	p	*Exact Value* R_n	$e^{-(k-1)np^2}$	*Lower Bound* (2)	*Lower Bound* (3)
10	2	0.01	0.9990	0.9990	0.9990	0.9990
		0.05	0.9764	0.9753	0.9755	0.9753
		0.10	0.9120	0.9048	0.9060	0.9044
		0.20	0.7053	0.6703	0.6733	0.6648
		0.50	0.1201	0.0821	0.0667	0.0563
		0.80	0.0013	0.0017	0.0001	0.0000
		0.90	0.0000	0.0003	0.0000	0.0000
	3	0.01	0.9981	0.9980	0.9980	0.9979
		0.05	0.9562	0.9512	0.9538	0.9500
		0.10	0.8485	0.8187	0.8345	0.8179
		0.20	0.5604	0.4493	0.5074	0.4573
		0.50	0.0449	0.0067	0.0179	0.0073
		0.80	0.0001	0.0000	0.0000	0.0000
		0.90	0.0000	0.0000	0.0000	0.0000
	5	0.01	0.9962	0.9960	0.9962	0.9955
		0.05	0.9222	0.9048	0.9202	0.8988
		0.10	0.7576	0.6703	0.7482	0.6704
		0.20	0.4094	0.2019	0.3847	0.2380
		0.50	0.0156	0.0000	0.0104	0.0006
		0.80	0.0000	0.0000	0.0000	0.0000

n	k	p	*Exact Value* R_n	$e^{-(k-1)np^2}$	*Lower Bound* (2)	*Lower Bound* (3)
100	2	0.001	0.9999	0.9999	0.9999	0.9999
		0.002	0.9996	0.9996	0.9996	0.9996
		0.005	0.9975	0.9975	0.9975	0.9975
		0.009	0.9920	0.9919	0.9919	0.9919
		0.010	0.9901	0.9900	0.9901	0.9901
		0.050	0.7875	0.7788	0.7787	0.7786
		0.100	0.3981	0.3679	0.3667	0.3660
		0.200	0.0304	0.0183	0.0171	0.0169
		0.500	0.0000	0.0000	0.0000	0.0000
	10	0.001	0.9991	0.9991	0.9991	0.9991
		0.002	0.9965	0.9964	0.9964	0.9963
		0.005	0.9791	0.9778	0.9783	0.9773
		0.009	0.9366	0.9297	0.9328	0.9295
		0.010	0.9232	0.9139	0.9181	0.9140
		0.050	0.2400	0.1054	0.1665	0.1441
		0.100	0.0128	0.0001	0.0025	0.0014
		0.200	0.0000	0.0000	0.0000	0.0000
		0.500	0.0000	0.0000	0.0000	0.0000

TABLE 1 (continued)

n	k	p	*Exact Value* R_n	$e^{-(k-1)np^2}$	*Lower Bound* (2)	*Lower Bound* (3)
100	20	0.001	0.9982	0.9981	0.9981	0.9980
		0.002	0.9928	0.9924	0.9927	0.9919
		0.005	0.9591	0.9536	0.9567	0.9517
		0.009	0.8833	0.8574	0.8728	0.8567
		0.010	0.8607	0.8270	0.8471	0.8276
		0.050	0.1139	0.0087	0.0600	0.0321
		0.100	0.0025	0.0000	0.0004	0.0001
		0.200	0.0000	0.0000	0.0000	0.0000
		0.500	0.0000	0.0000	0.0000	0.0000

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AMS Classification Numbers: 05A15, 11B39, 60J10

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