ON DIOPHANTINE APPROXIMATION BELOW THE LAGRANGE CONSTANT

Edward B. Burger and Jonathan M. Todd
Department of Mathematics, Williams College, Williamstown, MA 01267
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1. INTRODUCTION

For an irrational real number α, the *Lagrange* (often called the *Markoff*) *constant* for α, $\mu(\alpha)$, is defined by

$$\mu(\alpha) = \liminf_{q\to\infty} q\|\alpha q\|,$$

where $\| \ \|$ denotes the distance to the nearest integer function (see [3], although there the Lagrange constant is defined to be $\mu(\alpha)^{-1}$). Thus, for any c, $0 < c < \mu(\alpha)$, it follows that there are only finitely many positive integer solutions q to the inequality

$$q\|\alpha q\| < c. \tag{1.1}$$

We define $\lambda(\alpha)$ by $\lambda(\alpha) = \inf_{q>0} q\|\alpha q\|$.

Given α, two natural and fundamental problems are to compute $\lambda(\alpha)$ and, for a given c, $\lambda(\alpha) < c < \mu(\alpha)$, to explicitly determine the complete set of solutions to (1.1). In a series of three papers ([8], [9], [10]), Winley, Tognetti, and Van Ravenstein address these problems for the case in which α equals a *generalized golden ratio* φ_a, that is,

$$\alpha = \varphi_a = \frac{a+\sqrt{a^2+4}}{2},$$

where a is a positive integer. Not surprisingly, their solution involves generalized Fibonacci numbers. We write $\mathcal{F}_n = \mathcal{F}_n(a)$ for the n^{th} *generalized Fibonacci number*. That is, $\mathcal{F}_0 = 0$, $\mathcal{F}_1 = 1$ and, for $n > 1$, $\mathcal{F}_n = a\mathcal{F}_{n-1} + \mathcal{F}_{n-2}$. Using a well-known connection between $\mu(\alpha)$ and the continued fraction expansion of α (see [3]), one can, for these generalized golden ratios, explicitly compute $\mu(\varphi_a) = 1/\sqrt{a^2+4}$. Given this, we may state the main result of Winley et al. [10] as

Theorem 1: For a positive integer a, $\lambda(\varphi_a) = a/\varphi_a^2$. Moreover, for $\lambda(\varphi_a) < c \le 1/\sqrt{a^2+4}$, an integer $q > 0$ is a solution to

$$q\|\varphi_a q\| < c \tag{1.2}$$

if and only if $q = \mathcal{F}_{2m-1}$, where m is any positive integer satisfying

$$1 - c\sqrt{a^2+4} < \varphi_a^{-4m}. \tag{1.3}$$

The key to the proof of Theorem 1 is the observation that the numerators and denominators of the convergents of φ_a, which are the generalized Fibonacci numbers, enjoy a simple second-order recurrence relation.

In this paper we extend Theorem 1 to arbitrary real quadratic irrationals. Fundamental to our method is an important, but not widely known, result on the arithmetical structure of the sequences of numerators and denominators of the convergents of quadratic irrationals. In particular, each sequence may be partitioned into a finite number of simple second-order linear recurrence sequences all of which satisfy the same recurrence relation. We state this result explicitly as Theorem 3 in Section 2. A pleasant consequence of this result is a very simple method for computing the Lagrange constant for quadratic irrationals. This fundamental result is stated in Section 4 as Lemma 7.

As our generalization of inequality (1.3) for arbitrary quadratic irrationals requires the coefficients occurring in the recurrence relations given in Theorem 3, we postpone stating our main results, Theorems 5 and 6, until Section 3. However, below we illustrate our results with a simple extension of Theorem 1. We recall that the continued fraction expansion for φ_a is $[a, a, a, \ldots] = [\overline{a}]$. Thus, it is natural to next examine quadratic irrationals having a purely periodic continued fraction expansion of period length 2. In particular, we consider

$$\alpha(a,b) = \frac{ab + \sqrt{a^2b^2 + 4ab}}{2b} = \left[\overline{a, b}\right].$$

It follows by either a direct calculation or an application of Lemma 7, that

$$\mu(\alpha(a,b)) = \frac{\min\{a, b\}}{b(\alpha(a,b) - \overline{\alpha}(a,b))},$$

where $\overline{\alpha}(a,b)$ denotes the algebraic conjugate of $\alpha(a,b)$. If we let p_n/q_n be the n^{th} convergent of $\alpha(a,b)$, then the following is a special case of Theorem 5.

Theorem 2: For positive integers a and b, let $\alpha = \alpha(a,b)$. Then $\lambda(\alpha) = \min\{b^2\overline{\alpha} + b, \mu(\alpha)\}$. Moreover, for $\lambda(\alpha) < c < \mu(\alpha)$, an integer $q > 0$ is a solution to $q\|\alpha q\| < c$ if and only if $q = q_{2n+1}$, where $n \geq 0$ is any integer satisfying

$$((ab+2)(1+b\overline{\alpha}) - 1)^n < ((ab+2)(1+b\alpha) - 1)(1 - c(\alpha - \overline{\alpha})).$$

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2. RECURRENCE SEQUENCES AND QUADRATIC IRRATIONALS

For a real number α, we denote its simple continued fraction expansion by $[a_0, a_1, \ldots]$. We define the sequence of *convergents,* p_n/q_n, $n = 1, 2, \ldots$, by $p_n/q_n = [a_0, a_1, \ldots, a_n]$, where gcd $(p_n, q_n) = 1$. If we declare $p_{-2} = 0$, $p_{-1} = 1$, and $q_{-2} = 1$, $q_{-1} = 0$, then it follows that, for all $n \geq 0$, $p_n = a_n p_{n-1} + p_{n-2}$ and $q_n = a_n q_{n-1} + q_{n-2}$. By the well-known result of Lagrange, $\alpha \in \mathbb{R}$ is a quadratic irrational if and only if the continued fraction expansion for α is eventually periodic (see [4]). For the remainder of this paper, α is assumed to be a real quadratic irrational and thus we may denote its continued fraction as

$$\alpha = \left[a_0, a_1, \ldots, a_{k-1}, \overline{a_k, a_{k+1}, \ldots, a_{k+T-1}}\right].$$

Hence, for each t, $0 \leq t \leq T-1$, we have

$$p_{Tn+t+k} = a_{t+k} p_{Tn+t+k-1} + p_{Tn+t+k-2} \quad \text{and} \quad q_{Tn+t+k} = a_{t+k} q_{Tn+t+k-1} + q_{Tn+t+k-2}$$

for all $n = 0, 1, 2, \ldots$. We will require the following result which shows that the sequences $\{p_n\}$ and $\{q_n\}$ may be partitioned into T simple second-order linear recurrence sequences all satisfying the same recurrence relation. Such a result was also noted by Cusick [2] and again by van der Poorten [7] (see the related work of Kiss [5]), and is of some independent interest (see [1]).

Theorem 3: If

$$\alpha = [a_0, a_1, \dots, a_{k-1}, \overline{a_k, a_{k+1}, \dots, a_{k+T-1}}] \quad \text{and} \quad \mathbf{P}(\alpha) = [\overline{a_k, a_{k+1}, \dots, a_{k+T-1}}],$$

then, for each t, $0 \le t \le T-1$,

$$\begin{aligned} p_{Tn+t+k} &= \omega(\alpha) p_{T(n-1)+t+k} + (-1)^{T+1} p_{T(n-2)+t+k}, \\ q_{Tn+t+k} &= \omega(\alpha) q_{T(n-1)+t+k} + (-1)^{T+1} q_{T(n-2)+t+k}, \end{aligned} \tag{2.1}$$

for all $n = 2, 3, \dots$, where the constant $\omega(\alpha) = p^*_{T-1} + q^*_{T-2}$ and p^*_n / q^*_n denotes the n^{th} convergent of $\mathbf{P}(\alpha)$. Furthermore, if $\tau_1 = \mathbf{P}(\alpha)$ and $\tau_2 = \overline{\mathbf{P}(\alpha)}$, then $\tau_1 > 1$, $\tau_1 \tau_2 = (-1)^T$ and, for each t, $0 \le t \le T-1$, there exist real numbers u_t, v_t, r_t, s_t, with $r_t > 0$, such that $p_{Tn+t+k} = u_t \tau_1^n + v_t \tau_2^n$ and $q_{Tn+t+k} = r_t \tau_1^n + s_t \tau_2^n$ for all $n = 0, 1, 2, \dots$.

As it is difficult to find a proof of Theorem 3 in the literature, we include one here. We begin with the following elementary but useful lemma from linear algebra. The authors wish to thank the referee for suggesting the following elegant proof of Lemma 4.

Lemma 4: Let A, B, C, D be nonnegative integers satisfying $AD - BC = (-1)^T$ for some fixed integer T. If the sequences of integers $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ are defined by

$$\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^n,$$

then each of the four sequences, $\{a_n\}, \{b_n\}, \{c_n\}$, and $\{d_n\}$, satisfies the same second-order linear recurrence relation. In particular,

$$\begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix} = (A+D) \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} + (-1)^{T+1} \begin{pmatrix} a_{n-1} & b_{n-1} \\ c_{n-1} & d_{n-1} \end{pmatrix} \tag{2.2}$$

for $n \ge 2$,

Proof: If we write

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

then we note that the characteristic polynomial associated with M is given by

$$\det(M - \mathbf{1}_2 x) = x^2 - (A+D)x + AD - BC,$$

where $\mathbf{1}_2$ denotes the 2×2 identity matrix. By the Cayley-Hamilton Theorem, a matrix is a zero of its associated characteristic polynomial. Specifically, we have

$$M^2 = (A+D)M - (AD-BC)\mathbf{1}_2.$$

Multiplying the previous identity by M^{n-1} yields

$$M^{n+1} = (A+D)M^n - (AD-BC)M^{n-1}.$$

In view of our assumption that $AD - BC = (-1)^T$, the above equality becomes

$$M^{n+1} = (A+D)M^n + (-1)^{T+1}M^{n-1},$$

which completes the proof of the lemma.

Proof of Theorem 3: It is enough to prove that the identities in (2.1) hold, as the subsequent assertions of the theorem follow from (2.1) and the basic theory of linear recurrences. We first prove (2.1) in the case when $t = 0$ and then indicate how to modify the argument for $t \geq 1$.

By a well-known correspondence between partial quotients and convergents, if $p_n/q_n = [a_0, a_1, \ldots, a_n]$, then

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix},$$

(see, for example, [6]). Thus, given that $\mathbf{P}(\alpha) = [\overline{a_k, a_{k+1}, \ldots, a_{k+T-1}}]$, we have

$$\begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_{k+1} & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_{k+T-1} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p^*_{T-1} & p^*_{T-2} \\ q^*_{T-1} & q^*_{T-2} \end{pmatrix}, \tag{2.3}$$

where we recall that p^*_n/q^*_n denotes the n^{th} convergent of $\mathbf{P}(\alpha)$. Hence, as $\mathbf{P}(\alpha)$ has a purely periodic continued fraction expansion of period length T, we see that, for all $n \geq 1$,

$$\begin{pmatrix} p^*_{Tn-1} & p^*_{Tn-2} \\ q^*_{Tn-1} & q^*_{Tn-2} \end{pmatrix} = \begin{pmatrix} p^*_{T-1} & p^*_{T-2} \\ q^*_{T-1} & q^*_{T-2} \end{pmatrix}^n.$$

Taking determinants of both sides of (2.3), we see $p^*_{T-1}q^*_{T-2} - p^*_{T-2}q^*_{T-1} = (-1)^T$. Therefore, we may apply Lemma 4 and deduce that, for all $n \geq 2$,

$$\begin{pmatrix} p^*_{Tn-1} & p^*_{Tn-2} \\ q^*_{Tn-1} & q^*_{Tn-2} \end{pmatrix} = \omega(\alpha)\begin{pmatrix} p^*_{T(n-1)-1} & p^*_{T(n-1)-2} \\ q^*_{T(n-1)-1} & q^*_{T(n-1)-2} \end{pmatrix} + (-1)^{T+1}\begin{pmatrix} p^*_{T(n-2)-1} & p^*_{T(n-2)-2} \\ q^*_{T(n-2)-1} & q^*_{T(n-2)-2} \end{pmatrix},$$

where $\omega(\alpha) = p^*_{T-1} + q^*_{T-2}$.

Finally, turning our attention to the partial quotients of α, we note that, for all $n \geq 1$,

$$\begin{aligned} \begin{pmatrix} p_{Tn+k} & p_{Tn+k-1} \\ q_{Tn+k} & q_{Tn+k-1} \end{pmatrix} &= \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_{k-1} & 1 \\ 1 & 0 \end{pmatrix}\left(\begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_{k+T-1} & 1 \\ 1 & 0 \end{pmatrix}\right)^n \\ &= \begin{pmatrix} p_{k-1} & p_{k-2} \\ q_{k-1} & q_{k-2} \end{pmatrix}\begin{pmatrix} p^*_{T-1} & p^*_{T-2} \\ q^*_{T-1} & q^*_{T-2} \end{pmatrix}^n \\ &= \begin{pmatrix} p_{k-1} & p_{k-2} \\ q_{k-1} & q_{k-2} \end{pmatrix}\begin{pmatrix} p^*_{Tn-1} & p^*_{Tn-2} \\ q^*_{Tn-1} & q^*_{Tn-2} \end{pmatrix}. \end{aligned}$$

Thus, the pair (p_{Tn+k}, q_{Tn+k}) is a nonsingular, linear transformation of (p^*_{Tn-1}, q^*_{Tn-1}). Hence, both p_{Tn+k} and q_{Tn+k} each must satisfy the same second-order linear recurrence enjoyed by p^*_{Tn-1} and q^*_{Tn-1}. Specifically, for all $n \geq 2$,

$$\begin{aligned} p_{Tn+k} &= \omega(\alpha)p_{T(n-1)+k} + (-1)^{T+1}p_{T(n-2)+k}, \\ q_{Tn+k} &= \omega(\alpha)q_{T(n-1)+k} + (-1)^{T+1}q_{T(n-2)+k}, \end{aligned}$$

which proves the theorem when $t = 0$.

For $t \geq 1$, we note that

$$\begin{pmatrix} p_{Tn+t+k} & p_{Tn+k+t-1} \\ q_{Tn+t+k} & q_{Tn+k+t-1} \end{pmatrix} = \begin{pmatrix} p_{k-1} & p_{k-2} \\ q_{k-1} & q_{k-2} \end{pmatrix} \begin{pmatrix} p^*_{Tn-1} & p^*_{Tn-2} \\ q^*_{Tn-1} & q^*_{Tn-2} \end{pmatrix} \begin{pmatrix} p^*_{t-1} & p^*_{t-2} \\ q^*_{t-1} & q^*_{t-2} \end{pmatrix}.$$

Since both the first and third matrices appearing on the right-hand side of this identity are independent of n, we see that p_{Tn+t+k} and q_{Tn+t+k} are each linear combinations of p^*_{Tn-1}, p^*_{Tn-2}, q^*_{Tn-1}, and q^*_{Tn-2}. Thus, p_{Tn+t+k} and q_{Tn+t+k} must satisfy the same linear recurrence as p^*_{Tn-1}, p^*_{Tn-2}, q^*_{Tn-1}, and q^*_{Tn-2}. This fact establishes (2.1) for any t, $0 \leq t \leq T-1$, and completes the proof.

3. OUR MAIN RESULTS ON DIOPHANTINE APPROXIMATION

Given α as in Theorem 3, it will be convenient to define several new but natural constants that will allow us explicitly to determine $\lambda(\alpha)$. For each t, $0 \leq t \leq T-1$, we let $d_t = r_t v_t - s_t u_t$ and define

$$\lambda_t(\alpha) = \begin{cases} |d_t|\left(1+\frac{s_t}{r_t}\right), & \text{if } s_t < 0; \\ |d_t|, & \text{if } s_t > 0 \text{ and } T \text{ is even;} \\ |d_t|\left(1-\frac{s_t}{r_t}\tau_2^2\right), & \text{if } s_t > 0 \text{ and } T \text{ is odd.} \end{cases}$$

We now state our main result in the case when the continued fraction expansion for α is purely periodic.

Theorem 5: Suppose that $\alpha = [\overline{a_0, a_1, \ldots, a_{T-1}}]$; r_t and s_t are as in Theorem 3, and d_t and $\lambda_t(\alpha)$ are as defined above. Then $\lambda(\alpha) = \min\{\lambda_t(\alpha) : 0 \geq t \leq T-1\}$. Moreover, for $\lambda(\alpha) < c < \mu(\alpha)$, an integer $q > 0$ is a solution to

$$q\|\alpha q\| < c \tag{3.1}$$

if and only if $q = q_{Tn+t}$, where $0 \leq t \leq T-1$, $(-1)^{Tn} s_t \leq 0$, $\lambda_t(\alpha) < c$, and $n \geq 0$ satisfies

$$\frac{r_t}{|s_t|}\left(1 - \frac{c}{|d_t|}\right) < \overline{\alpha}^{2n}. \tag{3.2}$$

We remark that upon first inspection it may appear undesirable to have n occur in the bound $(-1)^{Tn} s_t \leq 0$. However, as T and t are known, it is only the *parity* of n that is necessary in computing this inequality. Hence, given c and t, one needs to find all even integers n that satisfy the conditions of the theorem and then all such odd integers. That is, implicit in the inequalities of Theorem 5 are the cases of n even and n odd.

Plainly, if the continued fraction expansion for the quadratic irrational α is not purely periodic, then there is no control on the size of the partial quotients occurring before the period; thus, one must examine each of the associated convergents individually. In particular, if

$$\alpha = [a_0, a_1, \ldots, a_{k-1}, \overline{a_k, a_{k+1}, \ldots, a_{k+T-1}}],$$

then there may be solutions to (3.1) among the first k convergents. With this unavoidable possibility understood, one has

Theorem 6: Suppose that $\alpha = [a_0, a_1, \ldots, a_{k-1}, \overline{a_k, a_{k+1}, \ldots, a_{k+T-1}}]$; $\mathbf{P}(\alpha)$, r_t, s_t are all as in Theorem 3, and d_t and $\lambda_t(\alpha)$ are as defined above. Let $\lambda_*(\alpha) = \min\{\lambda_t(\alpha) : 0 \le t \le T-1\}$. Then

$$\lambda(\alpha) = \min\{\lambda_*(\alpha), q_n\|\alpha q_n\| : 0 \le n < k\}.$$

Moreover, for $\lambda_*(\alpha) < c < \mu(\alpha)$, an integer $q \ge q_k$ is a solution to

$$q\|\alpha q\| < c \tag{3.3}$$

if and only if $q = q_{Tn+t+k}$, where $0 \le t \le T-1$, $(-1)^{Tn}s_t \le 0$, $\lambda_t(\alpha) < c$, and $n \ge 0$ satisfies

$$\frac{r_t}{|s_t|}\left(1 - \frac{c}{|d_t|}\right) < \overline{\mathbf{P}(\alpha)}^{2n}.$$

As an illustration, we return to $\alpha = \varphi_a = [\overline{a}]$. In this case, we have $T = 1$ and may verify that

$$\tau_1 = \varphi_a, \quad \tau_2 = \overline{\varphi}_a, \quad r_0 = \frac{\varphi_a}{\sqrt{a^2+4}}, \quad s_0 = \frac{-\overline{\varphi}_a}{\sqrt{a^2+4}},$$

$$u_0 = \frac{a\varphi_a + 1}{\sqrt{a^2+4}}, \quad v_0 = \frac{-a\overline{\varphi}_a - 1}{\sqrt{a^2+4}}, \text{ and thus } d_0 = \frac{-1}{\sqrt{a^2+4}}.$$

As $s_0 > 0$ and T is odd, it follows that

$$\lambda(\varphi_a) = \lambda_0(\varphi_a) = \frac{1}{\sqrt{a^2+4}}\left(1 - \left(\frac{-a+\sqrt{a^2+4}}{a+\sqrt{a^2+4}}\right)\frac{1}{\varphi_a^2}\right) = \frac{\varphi_a^2 - 1}{\varphi_a^3} = \frac{a}{\varphi_a^2},$$

as was stated in Theorem 1. We assume now that $\lambda(\varphi_a) < c < \mu(\varphi_a)$. For q_n to be a solution to (3.1), we must have $n > 0$ odd. If we write $n = 2m-1$, then (3.2) becomes

$$\frac{\varphi_a}{\overline{\varphi}_a}\left(1 - c\sqrt{a^2+4}\right) < \overline{\varphi}_a^{4m-2}.$$

Noting that $\varphi_a\overline{\varphi}_a = -1$ and $\varphi_a / |\overline{\varphi}_a| = \varphi_a^2$, the previous inequality is seen to be equivalent to

$$1 - c\sqrt{a^2+4} < \varphi_a^{-4m}. \tag{3.4}$$

Therefore, all the solutions to (3.1) are given by $q = q_{2m-1}$, where $m > 0$ satisfies (3.4) which, in view of the fact that $q_{2m-1} = \mathcal{F}_{2m-1}$, yields the result of Winley, Tognetti, and Van Ravenstein.

An Illustrative Collection of Examples: We briefly consider various numbers α equivalent to $\frac{1+\sqrt{3}}{2} = [\overline{1, 2}]$. For all such numbers, it follows that $\mu(\alpha) = \frac{1}{2\sqrt{3}}$.

1. Let $\alpha = [\overline{1, 2}]$. It follows that $\lambda_0(\alpha) = \frac{1}{2\sqrt{3}} = \mu(\alpha) \approx .288$ and $\lambda_1(\alpha) = 4 - 2\sqrt{3} \approx .535$, so $\lambda(\alpha) = \mu(\alpha)$. Hence, there are no solutions to (3.1) for any c, $0 < c \le \mu(\alpha)$.

2. $\alpha = [\overline{2, 1}] = 1 + \sqrt{3}$. We find that $\lambda_0(\alpha) = \frac{1}{\sqrt{3}}$ and $\lambda_1(\alpha) = 2 - \sqrt{3} \approx .267$. Thus, there are no solutions to (3.1) for $0 < c \le 2 - \sqrt{3}$; and for $2 - \sqrt{3} < c < \mu(\alpha)$ we have that q_{2n+1} is a solution to (3.1) for all integers $n \ge 0$ satisfying $(7 + 4\sqrt{3})(1 - c2\sqrt{3}) < (7 - 4\sqrt{3})^n$.

3. Let $\alpha = [3, 3, \overline{2, 1}] = \frac{38 - \sqrt{3}}{11}$. For $t = 1$, we have $(-1)^{Tn}s_t = s_1 = \frac{30 - 17\sqrt{3}}{6} > 0$. Thus, for $t = 1$, q_{Tn+t+k} is not a solution to (3.3). For $t = 0$, we note that $\lambda_0(\alpha) = \frac{91 - 49\sqrt{3}}{11} \approx .557 > \mu(\alpha) > c$. Thus, there are no solutions to (3.3) in this case either. A straightforward calculation shows that

$q_0 = 1$ and $q_1 = 3$ are never solutions to (3.3). Hence, for this α, there are no solutions to (3.3) for any $c < \mu(\alpha)$.

4. Let $\alpha = [3, 3, \overline{1, 2}] = 5 - \sqrt{3}$. As in the previous example, $s_1 = \frac{33-19\sqrt{3}}{6} \approx .015 > 0$ so there are no solutions to (3.3) for $t = 1$. Here, $\lambda_1(\alpha) = \frac{1}{\sqrt{3}} \approx .577$, $\lambda_0(\alpha) = 28 - 16\sqrt{3} \approx .287$; hence, $\lambda_*(\alpha) = 28 - 16\sqrt{3}$, while $\lambda(\alpha) = 1\|\alpha 1\| = 2 - \sqrt{3} \approx .267$. Therefore, after a calculation, we conclude that solutions to (3.3) for $28 - 16\sqrt{3} < c < \mu(\alpha)$ are p_{2n+2} / q_{2n+2} for all integers $n \ge 0$ satisfying $(97 + 56\sqrt{3})(1 - c2\sqrt{3}) < (7 - 4\sqrt{3})^n$. We also note that $q_0 = 1$ is the only solution to (3.1) for $0 < c \le 28 - 16\sqrt{3}$.

4. THE PROOF OF THEOREMS 5 AND 6

Before proceeding with our proof, we recall some elementary facts from the theory of continued fractions (see [4] or [6] for details). For an irrational real number α, the convergents p_n / q_n satisfy

$$\alpha - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n(\alpha_{n+1} q_n + q_{n-1})}, \tag{4.1}$$

where $\alpha_n = [a_n, a_{n+1}, a_{n+2}, \ldots]$ is the n^{th} *complete quotient.* We recall Hurwitz's celebrated result that $\mu(\alpha) \le 1/\sqrt{5}$ and Legendre's theorem that any rational solution p/q to

$$\left|\alpha - \frac{p}{q}\right| \le \frac{1}{2q^2},$$

must be a convergent of α.

We will make use of the following basic lemma which may be of some independent interest.

Lemma 7: Suppose that $\alpha = \left[a_0, a_1, \ldots, a_{k-1}, \overline{a_k, a_{k+1}, \ldots, a_{k+T-1}}\right]$ and $\tau_1, \tau_2, u_t, v_t, r_t, s_t$ are all as in Theorem 3. For each $0 \le t \le T-1$, let $d_t = r_t v_t - s_t u_t$. Then $\alpha = u_t / r_t$ for each *t* and

$$\mu(\alpha) = \min_{0 \le t \le T-1} \{|d_t|\}.$$

Proof: By Theorem 3 we have, for each *t*,

$$\frac{p_{Tn+t+k}}{q_{Tn+t+k}} = \frac{u_t \tau_1^n + v_t \tau_2^n}{r_t \tau_1^n + s_t \tau_2^n} = \frac{u_t + v_t(\tau_2 / \tau_1)^n}{r_t + s_t(\tau_2 / \tau_1)^n}.$$

In view of the fact that $|\tau_2 / \tau_1| < 1$, as $n \to \infty$ the above identity becomes $\alpha = u_t / r_t$.

Next, we observe that

$$\mu(\alpha) = \liminf_{q \to \infty} q\|\alpha q\| = \lim_{n \to \infty} q_n |\alpha q_n - p_n| = \lim_{n \to \infty} q_{n+k} |\alpha q_{n+k} - p_{n+k}|.$$

Finally, the first part of this lemma, together with a simple calculation, reveals

$$q_{Tn+t+k} |\alpha q_{Tn+t+k} - p_{Tn+t+k}| = \left|\left(\frac{u_t}{r_t}\right)(r_t \tau_1^n + s_t \tau_2^n)^2 - (u_t \tau_1^n + v_t \tau_1^n)(r_t \tau_1^n + s_t \tau_2^n)\right|$$

$$= \left|(-1)^{Tn}(s_t u_t - r_t v_t) + \tau_2^{2n}\left(s_t^2\left(\frac{u_t}{r_t}\right) - s_t v_t\right)\right| = |d_t| \left|(-1)^{Tn+1} - \left(\frac{s_t}{r_t}\right)\tau_2^{2n}\right|.$$

Since $|\tau_2| < 1$, we have

$$\lim_{n\to\infty} q_{Tn+t+k} |\alpha q_{Tn+t+k} - p_{Tn+t+k}| = |d_t|;$$

hence,

$$\mu(\alpha) = \min_{0\le t\le T-1} \{|d_t|\},$$

as desired.

Proof of Theorems 5 and 6: In view of the results of Hurwitz and Legendre, it is clear that any solution q to (3.3) for $0 < c < \mu(\alpha)$ must be a denominator of a convergent. We thus consider an arbitrary convergent p_m / q_m and, for a fixed t, $0 \le t \le T-1$, we examine all m such that $m-k \equiv t \bmod T$. That is, we write $m = Tn+t+k$.

Plainly, q_{Tn+t+k} is a solution of (3.3) if and only if

$$0 < \left| \alpha - \frac{p_{Tn+t+k}}{q_{Tn+t+k}} \right| < \frac{c}{q^2_{Tn+t+k}}.$$

In view of Theorem 3, identity (4.1), and the fact that $\alpha = u_t / r_t$, the above is equivalent to

$$0 < (-1)^{Tn+t+k} d_t \left((-1)^{Tn+1} - \frac{s_t}{r_t} \tau_2^{2n} \right) < c,$$

which may be simplified to

$$0 < (-1)^{t+k+1} d_t \left(1 + (-1)^{Tn} \frac{s_t}{r_t} \tau_2^{2n} \right) < c. \tag{4.2}$$

We note that the left side of the inequality (4.2) holds for all choices of n. Therefore, if we let n approach infinity, then as $|\tau_2| < 1$, we conclude that $(-1)^{t+k+1} d_t > 0$. Thus, (4.2) becomes

$$0 < |d_t| \left(1 + (-1)^{Tn} \frac{s_t}{r_t} \tau_2^{2n} \right) < c.$$

The right side of this inequality holds if and only if

$$(-1)^{Tn} \frac{s_t}{r_t} \tau_2^{2n} < \frac{c}{|d_t|} - 1. \tag{4.3}$$

Moreover, as $c < \mu(\alpha)$, Lemma 6 reveals $c < |d_t|$. Thus, the upper bound in (4.3) is negative. Since both τ_2^{2n} and r_t are positive, there are no solutions to (4.3) if $(-1)^{Tn} s_t > 0$. Thus, n is a solution to (4.3) if and only if $(-1)^{Tn} s_t \le 0$ and n satisfies (3.2).

Finally, we show that, for each t, $0 \le t \le T-1$, there are no solutions $q \ge q_k$ to (3.3) for $c \le \lambda_t(\alpha)$. As we have already seen,

$$q_{Tn+t+k} \|\alpha q_{Tn+t+k}\| = |d_t| \left(1 + (-1)^{Tn} \frac{s_t}{r_t} \tau_2^{2n} \right). \tag{4.4}$$

If $s_t < 0$, then as $0 < \tau_2^2 < 1$, (4.4) is minimized when $n = 0$. If $s_t > 0$ and T is even, the infimum of (4.4), $|d_t|$, is approached from above as $n \to \infty$. Finally, if $s_t > 0$ and T is odd, it is easy to see that (4.4) is minimized when $n = 1$, thus giving a minimum value of

$$|d_t|\left(1-\frac{s_t}{r_t}\tau_2^2\right).$$

These observations yield

$$\lambda_t(\alpha)=\inf_{n\geq 0}\{q_{Tn+t+k}\|\alpha q_{Tn+t+k}\|\}.$$

Hence, it follows that $\lambda(\alpha)=\min\{\lambda_*(\alpha), q_n\|\alpha q_n\|: 0\leq n<k\}$, and there are no solutions to (3.3) for $c\leq\lambda_t(\alpha)$ and $q\geq q_k$. This observation completes the proof.

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