ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Missouri State University, 800 University Drive, Maryville, MO 64468.

Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column. Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-895 ***Proposed by Indulis Strazdins, Riga Tech. Univ., Latvia***

Find a recurrence for F_{n^2}.

B-896 ***Proposed by Andrew Cusumano, Great Neck, NY***

Find an integer k such that the expression $F_n^4 + 2F_n^3F_{n+1} + kF_n^2F_{n+1}^2 - 2F_nF_{n+1}^3 + F_{n+1}^4$ is a constant independent of n.

B-897 ***Proposed by Brian D. Beasley, Presbyterian College, Clinton, SC***

Define $\langle a_n \rangle$ by $a_{n+3} = 2a_{n+2} + 2a_{n+1} - a_n$ for $n \geq 0$ with initial conditions $a_0 = 4$, $a_1 = 2$, and $a_2 = 10$. Express a_n in terms of Fibonacci and/or Lucas numbers.

B-898 ***Proposed by Alexandru Lupaş, Sibiu, Romania***

Evaluate

$$\sum_{k=0}^{s} (-1)^{(n-1)(s-k)} \binom{2s+1}{s-k} F_{n(2k+1)}.$$

B-899 ***Proposed by David M. Bloom, Brooklyn College of CUNY, Brooklyn, NY***

In a sequence of coin tosses, a *single* is a term (H or T) that is not the same as any adjacent term. For example, in the sequence HHTHHHTH, the singles are the terms in positions 3, 7, and 8. Let $S(n, r)$ be the number of sequences of n coin tosses that contain exactly r singles. If $n \geq 0$ and p is a prime, find the value modulo p of $\frac{1}{2}S(n+p-1, p-1)$.

SOLUTIONS

A Recurrence for nF_n

B-879 ***Proposed by Mario DeNobili, Vaduz, Lichtenstein (Vol. 37, no. 3, August 1999)***

Let $\langle c_n \rangle$ be defined by the recurrence $c_{n+4} = 2c_{n+3} + c_{n+2} - 2c_{n+1} - c_n$ with initial conditions $c_0 = 0$, $c_1 = 1$, $c_2 = 2$, and $c_3 = 6$. Express c_n in terms of Fibonacci and/or Lucas numbers.

Solution by H.-J. Seiffert, Berlin, Germany

Let $\langle c_n \rangle$ satisfy the given recurrence, but with initial conditions $c_0 = 0$, $c_1 = a$, $c_2 = 2b$, and $c_3 = 3(a+b)$, where a and b are any fixed numbers. We shall prove that $c_n = nH_n$, where the sequence $\langle H_n \rangle$ satisfies the recurrence $H_{n+2} = H_{n+1} + H_n$ with initial values $H_1 = a$ and $H_2 = b$. Direct computation shows that the equation $c_n = nH_n$ holds for $n = 0, 1, 2$, and 3. Suppose that it is true for all $k \in \{0, 1, \ldots, n+3\}$, where $n \geq 0$. We then have

$$\begin{aligned} c_{n+4} &= 2c_{n+3} + c_{n+2} - 2c_{n+1} - c_n \\ &= 2(n+3)H_{n+3} + (n+2)H_{n+2} - 2(n+1)H_{n+1} - nH_n \\ &= (n+4)(H_{n+3} + H_{n+2}) + n(H_{n+3} - 2H_{n+1} - H_n) + 2(H_{n+3} - H_{n+2} - H_{n+1}) \\ &= (n+4)H_{n+4}; \end{aligned}$$

note that $H_{n+3} = H_{n+2} + H_{n+1} = 2H_{n+1} + H_n$. This completes the induction proof. It can be shown that this equation holds for negative n as well.

To solve the present proposal, take $a = b = 1$. Then $c_n = nF_n$. With $a = 1$ and $b = 3$, we have $\langle H_n \rangle = \langle L_n \rangle$, and therefore, $c_n = nL_n$.

Solutions also received by Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, N. Gauthier, Joe Howard, Harris Kwong, James A. Sellers, Indulis Strazdins, and the proposer.

A Sum for F_{2m+2}

B-880 ***Proposed by A. J. Stam, Winsum, The Netherlands (Vol. 37, no. 3, August 1999)***

Express

$$\sum_{2i \leq m} \binom{m-i}{i} (-1)^i 3^{m-2i}$$

in terms of Fibonacci and/or Lucas numbers.

Solution by Harris Kwong, SUNY College at Fredonia, NY

Denote the given sum by s_m. The generating function of the sequence $\langle s_m \rangle$ is

$$\begin{aligned} \sum_{m=0}^{\infty} s_m z^m &= \sum_{m=0}^{\infty} \sum_{2i \leq m} \binom{m-i}{i} (-1)^i 3^{m-2i} z^m = \sum_{k=0}^{\infty} \sum_{2i \leq k+i} \binom{k}{i} (-1)^i 3^{k-i} z^{k+i} \\ &= \sum_{k=0}^{\infty} \sum_{i \leq k} \binom{k}{i} (-1)^i (3z)^{k-i} z^{2i} = \sum_{k=0}^{\infty} (3z - z^2)^k = \frac{1}{1 - 3z + z^2}. \end{aligned}$$

Since $1 - 3x^2 + x^4 = (1-x^2)^2 - x^2$, we find

$$\sum_{m=0}^{\infty} s_m x^{2m+2} = \frac{x^2}{(1-x^2)^2 - x^2} = \frac{1}{2}\left\{\frac{x}{1-x-x^2} - \frac{x}{1+x-x^2}\right\}$$
$$= \frac{1}{2}\left\{\sum_{n=0}^{\infty} F_n x^n + \sum_{n=0}^{\infty} F_n (-x)^n\right\} = \sum_{n=0}^{\infty} F_{2n} x^{2n}.$$

Therefore, $s_m = F_{2m+2}$.

Redmond showed that

$$\sum_{2i \le m} \binom{m-i}{i} (-1)^i (2x)^{m-2i} = U_m(x),$$

where $U_m(x)$ is the m^{th} *Chebyshev polynomial of the second kind.*

Cook noted that the problem is a slight variation of a problem posed by Mrs. William Squire in [1], solved by M. N. S. Swamy in [2], and further explored by H. W. Gould in [3].

References

1. Mrs. W. Squire. "Problem H-83." *The Fibonacci Quarterly* **4.1** (1966):57.
2. M. N. S. Swamy. "Solution of H-83." *The Fibonacci Quarterly* **6.1** (1968):54-55.
3. H. W. Gould. "A Fibonacci Formula of Lucas and Its Subsequent Manifestations and Rediscoveries." *The Fibonacci Quarterly* **15.1** (1977):25-29.

Seiffert reported that the identities

$$\sum_{2i \le m} \binom{m-i}{i} (-xy)^i (x+y)^{m-2i} = \frac{x^{m+1} - y^{m+1}}{x-y}, \ m \ge 0,$$

and

$$\sum_{2i \le m} \frac{m}{m-i} \binom{m-i}{i} (-xy)^i (x+y)^{m-2i} = x^m + y^m, \ m \ge 1,$$

are due to E. Lucas (Théorie des Nombres [Paris: Blanchard, 1961], Ch. 18).

Solutions also received by Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Don Redmond, H.-J. Seiffert, Indulis Strazdins, and the proposer.

Diophantine Pair

B-881 ***Proposed by Mohammad K. Azarian, University of Evansville, IN (Vol. 37, no. 3, August 1999)***

Consider the two equations

$$\sum_{i=1}^{n} L_i x_i = F_{n+3} \quad \text{and} \quad \sum_{i=1}^{n} L_i y_i = L_3 - F_{n+1}.$$

Show that the number of positive integer solutions of the first equation is equal to the number of nonnegative integer solutions of the second equation.

Solution by L. A. G. Dresel, Reading, England

We have $(L_1 + L_2 + \cdots + L_n) = \Sigma(L_{j+2} - L_{j+1}) = L_{n+2} - L_2$. Subtracting this from the first of the given equations and using the identity $L_{n+2} = F_{n+3} + F_{n+1}$, we obtain $\Sigma L_i(x_i - 1) = L_2 - F_{n+1}$. Putting $y_i = x_i - 1$, it follows that y_i is a nonnegative integer whenever x_i is a positive integer. Similarly, starting with the second of the given equations, we can obtain the first given equation.

Thus, the two given equations are equivalent and have the same number of solutions of the specified kinds.

Generalization by Harris Kwong, SUNY College at Fredonia, NY

We give a generalization. Let $\langle u_n \rangle$ be a sequence that satisfies the recurrence $u_n = u_{n-1} + u_{n-2}$. It can be proved, for instance, by induction, that $\sum_{i=1}^{n} u_i = u_{n+2} - u_2$. Let $\langle v_n \rangle$ be another sequence. Consider the equations

$$\sum_{i=1}^{n} u_i x_i = v_{n+3} \quad \text{and} \quad \sum_{i=1}^{n} u_i y_i = v_{n+3} - u_{n+2} + u_2 .$$

Every positive integer solution $(a_1, a_2, \ldots, a_n)$ of the first equation yields a nonnegative solution $(a_1 - 1, a_2 - 1, \ldots, a_n - 1)$ of the second equation. Conversely, any nonnegative solution $(b_1, b_2, \ldots, b_n)$ of the second equation leads to a positive solution $(b_1 + 1, b_2 + 1, \ldots, b_n + 1)$ of the first equation. Therefore, the positive solutions of the first equation and the nonnegative solutions of the second equation are in one-to-one correspondence. In particular, when $u_n = L_n$ and $v_n = F_n$, the two equations reduce to the ones in the problem statement, because $L_{n+2} = F_{n+3} + F_{n+1}$.

Solutions also received by Paul S. Bruckman, H.-J. Seiffert, Indulis Strazdins, and the proposer.

A Multiple of F_{n+1}

B-882 *Proposed by A. J. Stam, Winsum, The Netherlands (Vol. 37, no. 3, August 1999)*

Suppose the sequence $\langle A_n \rangle$ satisfies the recurrence $A_n = A_{n-1} + A_{n-2}$. Let

$$B_n = \sum_{k=0}^{n} (-1)^k A_{n-2k} .$$

Prove that $B_n = A_0 F_{n+1}$ for all nonnegative integers n.

Solution by H.-J. Seiffert, Berlin, Germany

Let x be any complex number and suppose that the sequence $\langle A_n(x) \rangle$ satisfies the recurrence $A_n(x) = xA_{n-1}(x) + A_{n-2}(x)$. Define

$$B_n(x) = \sum_{k=0}^{n} (-1)^k A_{n-2k}(x).$$

We shall prove that $B_n(x) = A_0(x) F_{n+1}(x)$ for all nonnegative integers n, where $\langle F_n(x) \rangle$ denotes the sequence of Fibonacci polynomials which satisfies the same recurrence as $\langle A_n(x) \rangle$, but with given initial conditions $F_0(x) = 0$ and $F_1(x) = 1$.

If $n \geq 2$, then

$$\begin{aligned} B_n(x) &= x\sum_{k=0}^{n} (-1)^k A_{n-1-2k}(x) + \sum_{k=0}^{n} (-1)^k A_{n-2-2k}(x) \\ &= x\sum_{k=0}^{n-1} (-1)^k A_{n-1-2k}(x) + \sum_{k=0}^{n-2} (-1)^k A_{n-2-2k}(x) \\ &\quad + (-1)^n (xA_{-n-1}(x) + A_{-n-2}(x) - A_{-n}(x)) \end{aligned}$$

or $B_n(x) = xB_{n-1}(x) + B_{n-2}(x)$. Since $B_0(x) = A_0(x) = A_0(x)F_1(x)$ and $B_1(x) = A_1(x) - A_{-1}(x) = xA_0(x) = A_0(x)F_2(x)$, the desired equation now follows by a simple induction argument.

The proposal's result is obtained when taking $x = 1$.

Solutions also received by Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, N. Gauthier, Pentti Haukkanen, Joe Howard, Harris Kwong, Don Redmond, James A. Sellers, Indulis Strazdins, and the proposer.

Property of a Periodic Sequence

B-883 *Proposed by L. A. G. Dresel, Reading, England*
(Vol. 37, no. 3, August 1999)

Let $\langle f_n \rangle$ be the Fibonacci sequence F_n modulo p, where p is a prime, so that we have $f_n \equiv F_n \pmod{p}$ and $0 \le f_n < p$ for all $n \ge 0$. The sequence $\langle f_n \rangle$ is known to be periodic. Prove that, for a given integer c in the range $0 \le c < p$, there can be at most four values of n for which $f_n = c$ within any one cycle of this period.

Solution by the proposer

From the identities $F_{n+1} + F_{n-1} = L_n$ and $F_{n+1} - F_{n-1} = F_n$, we obtain $2F_{n+1} = L_n + F_n$, and we also have $L_n^2 = 5(F_n)^2 + 4(-1)^n$. We shall assume first that $p \ne 2$, and that $F_n \equiv c \pmod{p}$ for some even value of n. Then it follows that $L_n \equiv \pm\sqrt{(5c^2+4)}$ and $2F_{n+1} \equiv c \pm \sqrt{(5c^2+4)} \pmod{p}$; this gives two possible values for f_{n+1}, say s_1 and s_2. It is possible that we also have $f_n \equiv c$ occurring for some odd value of n, so that we have $f_{n+1} \equiv c \pm \sqrt{(5c^2-4)} \pmod{p}$, giving two further possible values s_3 and s_4, say. These values may not all be distinct, but clearly there are at most four different values of s which can follow c in the sequence $\langle f \rangle$. But if a given consecutive pair of values c, s were to occur a second time, the sequence $\langle f \rangle$ would repeat itself because of the recurrence relation. Hence, the value c can occur at most four times in $\langle f \rangle$ within one cycle of the period, namely, at most twice for an even value of n and at most twice for an odd value of n.

For the special case $p = 2$, we see that a complete cycle is $\langle f \rangle \equiv 0, 1, 1 \pmod{2}$.

Corollary: Since there are only p values of c in the range $0 \le c \le p-1$, it follows that the period $K(p)$ of the sequence F_n modulo p satisfies $K(p) \le 4p$. In the special case $p = 5$, we do in fact obtain $K(5) = 20$.