TWO PROOFS OF FILIPPONI'S FORMULA FOR ODD-SUBSCRIPTED LUCAS NUMBERS

Alois Panholzer and Helmut Prodinger
Dept of Mathematics, University of the Witwatersrand
PO Wits, 2050 Johannesburg, South Africa
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In a recent paper [1], Filipponi presented, without proof, a formula for odd-subscripted Lucas numbers which can be equivalently rewritten as

$$L_{2n+1} = \sum_{0\le j\le \frac{n}{2}} (-1)^j \binom{n-j}{j} 3^{n-1-2j} \frac{4n-5j}{n-j}. \tag{1}$$

This form is better suited for our treatment, and we can observe that, for even n, the summand for $j=\frac{n}{2}$ is $(-1)^{n/2}$, explaining the extra term in [1]. The aim of this note is to give two distinct proofs of (1).

First proof of (1). We use the standard forms

$$G_n(w) = \sum_{0\le k\le n} \binom{n-k}{k} w^k. \tag{2}$$

They are studied very well in [2], and we have, for $n \ge 0$,

$$\sum_{0\le j\le \frac{n}{2}} (-1)^j \binom{n-j}{j} 3^{n-1-2j} \frac{4n-5j}{n-j} = 3^n G_n\left(-\tfrac{1}{9}\right) + 3^{n-1} G_{n-1}\left(-\tfrac{1}{9}\right).$$

In Exercise 7.34 of [2] we find also the generating function

$$G(z,w) = \sum_{n\ge 0} G_n(w) z^n = \frac{1}{1-z-wz^2}.$$

Hence,

$$\begin{aligned}
&\sum_{n\ge 0}\left(3^n G_n\left(-\tfrac{1}{9}\right) + 3^{n-1} G_{n-1}\left(-\tfrac{1}{9}\right)\right) z^n \\
&= \sum_{n\ge 0} G_n\left(-\tfrac{1}{9}\right)(3z)^n + z\sum_{n\ge 0} G_n\left(-\tfrac{1}{9}\right)(3z)^n \\
&= \frac{1+z}{1-3z+z^2}.
\end{aligned}$$

Since a trivial computation gives

$$\sum_{n\ge 0} L_{2n+1} z^n = \frac{1+z}{1-3z+z^2},$$

the proof is finished. □

We can even get a general formula for L_{sn+t} with nonnegative integers $0 \le t < s$; for that, we set up generating functions:

$$\begin{aligned} F_{s,t}(z) &= \sum_{n\geq 0} L_{sn+t} z^n = \alpha^t \sum_{n\geq 0} (\alpha^s z)^n + \beta^t \sum_{n\geq 0} (\beta^s z)^n = \alpha^t \frac{1}{1-\alpha^s z} + \beta^t \frac{1}{1-\beta^s z} \\ &= \frac{\alpha^t(1-\beta^s z) + \beta^t(1-\alpha^s z)}{(1-\alpha^s z)(1-\beta^s z)} = \frac{\alpha^t + \beta^t - ((-1)^t \alpha^{s-t} + (-1)^t \beta^{s-t}) z}{1-(\alpha^s+\beta^s)z + (-1)^s z^2} \\ &= \frac{L_t - (-1)^t L_{s-t} z}{1 - L_s z + (-1)^s z^2}. \end{aligned} \tag{3}$$

By using (2), (3) can be written as

$$F_{s,t}(z) = (L_t - (-1)^t L_{s-t} z) G\left(L_s z, \frac{(-1)^{s-1}}{L_s^2}\right)$$

and, therefore, we get the formula:

$$\begin{aligned} L_{sn+t} &= L_t \sum_{0\leq k\leq n} \binom{n-k}{k} (-1)^{(s-1)k} L_s^{n-2k} \\ &\quad - (-1)^t L_{s-t} \sum_{0\leq k\leq n-1} \binom{n-1-k}{k} (-1)^{(s-1)k} L_s^{n-1-2k}. \end{aligned} \tag{4}$$

We do not know whether this formula is new, but it is easy to prove and generates infinitely many "Filipponi formulas."

Second proof of (1). For the second (mechanical) proof ("Zeilberger's algorithm"), we note the following (see [3] and [2] for the underlying theory): Set

$$f(n,k) := (-1)^k \binom{n-k}{k} 3^{n-1-2k} \frac{4n-5k}{n-k} \quad \text{and} \quad F(n) := \sum_k f(n,k).$$

Furthermore, set

$$g(n,k) := -\frac{9k(n-k)(4n-5k+5)}{(n-2k+2)(n-2k+1)(4n-5k)} f(n,k),$$

then

$$f(n+2,k) - 3f(n+1,k) + f(n,k) = g(n,k+1) - g(n,k)$$

(check!!), thus we get, on summing over k, $F(n+2) - 3F(n+1) + F(n) = 0$. Since the odd-subscripted Lucas numbers also satisfy the recursion $L_{2n+5} - 3L_{2n+3} + L_{2n+1} = 0$ and two initial values match as well, the proof is finished. □

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