DERIVATIVE SEQUENCES OF GENERALIZED JACOBSTHAL AND JACOBSTHAL-LUCAS POLYNOMIALS

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1. INTRODUCTION

In this note we define two sequences $\{J_{n,m}(x)\}$—the generalized Jacobsthal polynomials, and $\{j_{n,m}(x)\}$—the generalized Jacobsthal-Lucas polynomials, by the following recurrence relations:

$$J_{n,m}(x)=J_{n-1,m}(x)+2xJ_{n-m,m}(x),\quad n\geq m, \tag{1.1}$$

with starting polynomials $J_{0,m}(x)=0$, $J_{n,m}(x)=1$, $n=1,2,\ldots,m-1$, and

$$j_{n,m}(x)=j_{n-1,m}(x)+2xj_{n-m,m}(x),\quad n\geq m, \tag{1.2}$$

with starting polynomials $j_{0,m}(x)=2$, $j_{n,m}(x)=1$, $n=1,2,\ldots,m-1$.

For $m=2$, these polynomials are studied in [1], [2], and [3].

From (1.1) and (1.2), using the standard method, we find that the polynomials $\{J_{n,m}(x)\}$ and $\{j_{n,m}(x)\}$ have, respectively, the following generating functions:

$$F(x,t)=(1-t-2xt^m)^{-1}=\sum_{n=1}^{\infty}J_{n,m}(x)t^{n-1} \tag{1.3}$$

and

$$G(x,t)=\frac{1+4xt^{m-1}}{1-t-2xt^m}=\sum_{n=1}^{\infty}j_{n,m}(x)t^{n-1}. \tag{1.4}$$

From (1.3) and (1.4), we find the following explicit representations for the polynomials $\{J_{n,m}(x)\}$ and $\{j_{n,m}(x)\}$:

$$J_{n,m}(x)=\sum_{k=0}^{[(n-1)/m]}\binom{n-1-(m-1)k}{k}(2x)^k \tag{1.5}$$

and

$$j_{n,m}(x)=\sum_{k=0}^{[n/m]}\frac{n-(m-2)k}{n-(m-1)k}\binom{n-(m-1)k}{k}(2x)^k. \tag{1.6}$$

Differentiating (1.5) and (1.6) with respect to x, we get

$$J_{n,m}^{(1)}(x)=\sum_{k=1}^{[(n-1)/m]}2k\binom{n-1-(m-1)k}{k}(2x)^{k-1} \tag{1.7}$$

and

$$j_{n,m}^{(1)}(x)=\sum_{k=1}^{[n/m]}2k\frac{n-(m-2)k}{n-(m-1)k}\binom{n-(m-1)k}{k}(2x)^{k-1}, \tag{1.8}$$

with

$$J_{n,m}^{(1)}(x)=j_{n,m}^{(1)}(x)=0,\quad n=0,1,\ldots,m-1. \tag{1.9}$$

For $x=1$ in (1.5), (1.6), (1.7), and (1.8), we have, respectively: $\{J_{n,m}(1)\}$—the generalized Jacobsthal numbers, $\{j_{n,m}(1)\}$—the generalized Jacobsthal-Lucas numbers, $\{J_{n,m}^{(1)}(1)\}$—the generalized Jacobsthal derivative sequence, and $\{j_{n,m}^{(1)}(1)\}$—the generalized Jacobsthal-Lucas derivative sequence.

The aim of this note is to study some characteristic properties of the sequences of numbers $\{J_{n,m}^{(1)}(1)\}$ and $\{j_{n,m}^{(1)}(1)\}$. We shall use the notations $H_{n,m}^1$ instead of $J_{n,m}^{(1)}(1)$ and $K_{n,m}^1$ instead of $j_{n,m}^{(1)}(1)$.

The first few members of the sequences $\{J_{n,m}(x)\}$, $\{J_{n,m}^{(1)}(x)\}$, and $\{H_{n,m}^1\}$ are presented in Table 1, and the first few members of the sequences $\{j_{n,m}(x)\}$, $\{j_{n,m}^{(1)}(x)\}$, and $\{K_{n,m}^1\}$ are given in Table 2.

TABLE 1

n	$J_{n,m}(x)$	$J_{n,m}^{(1)}(x)$	$H_{n,m}^1$
0	0	0	0
1	1	0	0
2	1	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$m-1$	1	0	0
m	1	0	0
$m+1$	$1+2x$	2	2
$m+2$	$1+4x$	4	4
$m+3$	$1+6x$	6	6
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2m-1$	$1+2(m-1)x$	$2(m-1)$	$2(m-1)$
$2m$	$1+2mx$	$2m$	$2m$
$2m+1$	$1+2(m+1)x+4x^2$	$2(m+1)+8x$	$2m+10$
$2m+2$	$1+2(m+2)x+12x^2$	$2(m+2)+24x$	$2m+28$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

TABLE 2

n	$j_{n,m}(x)$	$j_{n,m}^{(1)}(x)$	$K_{n,m}^1$
0	2	0	0
1	1	0	0
2	1	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$m-1$	1	0	0
m	$1+4x$	4	0
$m+1$	$1+6x$	6	0
$m+2$	$1+8x$	8	0
$m+3$	$1+10x$	10	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2m-1$	$1+2(m+1)x$	$2(m+1)$	$2(m+1)$
$2m$	$1+2(m+2)x+8x^2$	$2(m+2)+16x$	$2m+20$
$2m+1$	$1+2(m+3)x+20x^2$	$2(m+3)+40x$	$2m+46$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

From Table 1 and Table 2, we can prove by induction and (1.1) the following relation:

$$\begin{aligned} j_{n,m}(x) &= J_{n,m}(x) + 4xJ_{n+1-m,m}(x) \\ &= J_{n+1,m}(x) + 2xJ_{n+1-m,m}(x), \quad [\text{by } (1.1)]. \end{aligned} \tag{1.10}$$

Observe that the first equation in (1.10) is a direct consequence of (1.3) and (1.4).

2. SOME PROPERTIES OF $H^1_{n,m}$ AND $K^1_{n,m}$

Differentiating (1.3) and (1.4) with respect to x, we get the following generating functions, respectively:

$$\sum_{n=0}^{\infty} J^{(1)}_{n,m}(x)t^n = \frac{2t^{m+1}}{(1-t-2xt^m)^2} \tag{2.1}$$

and

$$\sum_{n=0}^{\infty} j^{(1)}_{n,m}(x)t^n = \frac{2t^m(2-t)}{(1-t-2xt^m)^2}. \tag{2.2}$$

Hence, for $x=1$ in (2.1) and (2.2), we get the generating functions for $H^1_{n,m}$ and $K^1_{n,m}$, respectively:

$$\sum_{n=0}^{\infty} H^1_{n,m}t^n = \frac{2t^{m+1}}{(1-t-2t^m)^2} \tag{2.3}$$

and

$$\sum_{n=0}^{\infty} K^1_{n,m}t^n = \frac{2t^m(2-t)}{(1-t-2t^m)^2}. \tag{2.4}$$

If we substitute $x=1$ in (1.1) and (1.2), we get the sequences of numbers $\{J_{n,m}\}$ and $\{j_{n,m}\}$, which satisfy the following relations:

$$j_{n,m} = J_{n,m} + 4J_{n+1-m,m} = J_{n+1,m} + 2J_{n+1-m,m} \quad [\text{by } (1.10)], \tag{2.5}$$

$$j_{n+1,m} + j_{n,m} = 3J_{n+1,m} + 4J_{n+2-m,m} - J_{n,m} \quad [\text{by } (2.5), (1.1)], \tag{2.6}$$

$$j_{n+1,m} - j_{n,m} = 4J_{n+2-m,m} + J_{n,m} - J_{n+1,m} \quad [\text{by } (2.5), (1.1)], \tag{2.7}$$

$$j_{n+1,m} - 2j_{n,m} = 4J_{n+2-m,m} + 2J_{n,m} - 3J_{n+1,m} \quad [\text{by } (2.5), (1.1)], \tag{2.8}$$

$$J_{n,m} + j_{n,m} = 2J_{n+1,m}. \tag{2.9}$$

For $m=2$, relations (2.5)-(2.9) yield the following relations:

$$\begin{aligned} &j_n = J_{n+1} + 2J_{n-1} && ((2.10) \text{ in } [2]), \\ &j_{n+1} + j_n = 3(J_{n+1} + J_n) && ((2.12) \text{ in } [2]), \\ &j_{n+1} - j_n = 5J_n - J_{n+1}, && \\ &j_{n+1} - 2j_n = 3(2J_n - J_{n+1}) && ((2.14) \text{ in } [2]), \\ &J_n + j_n = 2J_{n+1} && ((2.20) \text{ in } [2]), \end{aligned}$$

where $J_{n,2} = J_n$ and $j_{n,2} = j_n$.

Differentiating (1.1) and (1.2) with respect to x, and substituting $x=1$. we get the following recurrence relations:

$$H^1_{n,m} = H^1_{n-1,m} + 2H^1_{n-m,m} + 2J_{n-m,m}, \quad n \geq m, \tag{2.10}$$

with $H^1_{n,m} = 0$, $n = 0, 1, \ldots, m-1$ and

$$K^1_{n,m} = K^1_{n-1,m} + 2K^1_{n-m,m} + 2j_{n-m,m}, \quad n \geq m, \tag{2.11}$$

with $K^1_{n,m} = 0$, $n = 0, 1, \ldots, m-1$.

In a similar way, from (1.10), we get

$$K^1_{n,m} = H^1_{n,m} + 4H^1_{n+1-m,m} + 4J_{n+1-m,m}, \quad n \geq m-1. \tag{2.12}$$

For $m=2$, relations (2.10)-(2.12) become

$$H^1_{n+2} = H^1_{n+1} + 2H^1_n + 2J_n \quad ((3.3) \text{ in } [1]),$$

$$K^1_{n+2} = K^1_{n+1} + 2K^1_n + 2j_n \quad ((3.4) \text{ in } [1]),$$

$$K^1_{n+1} = H^1_{n+1} + 4H^1_n + 4J_n.$$

From (2.10) and (2.12), we get $K^1_{n,m} + H^1_{n,m} = 2H^1_{n+1,m}$.

For $m=2$, the last equality yields the known relation (3.8) in [1].

Again, from (2.10) and (2.12), we find

$$K^1_{n,m} - H^1_{n,m} = 2H^1_{n+1,m} - 2H^1_{n,m}. \tag{2.13}$$

For $m=2$, (2.13) becomes (3.9) in [1].

Theorem 2.1: The polynomials $\{J_{n,m}(x)\}$ and $\{j_{n,m}(x)\}$ satisfy the following relations, respectively,

$$\sum_{i=0}^{n} J_{i,m}(x) = \frac{J_{n+m,m}(x)-1}{2x} \tag{2.14}$$

and

$$\sum_{i=0}^{n} j_{i,m}(x) = \frac{j_{n+m,m}(x)-1}{2x}. \tag{2.15}$$

Proof: From (1.1) and (1.2), by induction on n, we can prove (2.14) and (2.15).

Corollary 2.1: For $m=2$ in (2.14) and (2.15), we get the known relations (2.7) and (2.8) in [2].

Theorem 2.2: The numbers $H^1_{i,m}$ and $K^1_{n,m}$ satisfy the following relations, respectively,

$$\sum_{i=0}^{n} H^1_{i,m} = 1/2(H^1_{n+m,m} - J_{n+m,m} + 1) \tag{2.16}$$

and

$$\sum_{i=0}^{n} K^1_{i,m} = 1/2(K^1_{n+m,m} - j_{n+m,m} + 1). \tag{2.17}$$

Proof: Differentiating (2.14) and (2.15), respectively, with respect to x, and substituting $x=1$, we get (2.16) and (2.17).

Corollary 2.2: For $m = 2$, from (2.16) and (2.17), we have

$$\sum_{i=0}^{n} H_i^1 = 1/2(H_{n+2}^1 - J_{n+2} + 1) \quad \text{and} \quad \sum_{i=0}^{n} K_i^1 = 1/2(K_{n+2}^1 - j_{n+2} + 1).$$

Furthermore, from (1.7), we get

$$H_{n+m,\,m}^1 + 2(m-1)H_{n,\,m}^1 - 2nJ_{n,\,m}. \tag{2.18}$$

For $m = 2$ in (2.18), we have ((3.6′) in [1]), $H_{n+2}^1 + 2H_n^1 = 2nJ_n$.

In a similar way, from (1.8), we get

$$K_{n,\,m}^1 = 2(n+2-m)J_{n+1-m,\,m} - 2(m-2)H_{n+1-m,\,m}^1. \tag{2.19}$$

For $m = 2$ in (2.19), we obtain ((2.4) in [1]), $K_n^1 = 2nJ_{n-1}$.

GENERALIZATION

Differentiating (1.1), (1.2), and (1.10) k times with respect to x, we get

$$J_{n,\,m}^{(k)}(x) = J_{n-1,\,m}^{(k)}(x) + 2kJ_{n-m,\,m}^{(k-1)}(x) + 2xJ_{n-m,\,m}^{(k)}(x), \quad k \geq 1,\ n \geq m,$$

$$j_{n,\,m}^{(k)}(x) = j_{n-1,\,m}^{(k)}(x) + 2kj_{n-m,\,m}^{(k-1)}(x) + 2xj_{n-m,\,m}^{(k)}(x), \quad k \geq 1,\ n \geq m,$$

$$j_{n,\,m}^{(k)}(x) = J_{n-1,\,m}^{(k)}(x) + 4kJ_{n+1-m,\,m}^{(k-1)}(x) + 4xJ_{n+1-m,\,m}^{(k)}(x), \quad k \geq 1,\ n \geq m,$$

respectively.

From the last equalities, using the notations $J_{n,\,m}^{(k)}(1) \equiv H_{n,\,m}^k$ and $j_{n,\,m}^{(k)}(1) \equiv K_{n,\,m}^k$, we can prove the following relations:

$$H_{n,\,m}^k = H_{n-1,\,m}^k + 2kH_{n-m,\,m}^{k-1} + 2H_{n-m,\,m}^k, \quad k \geq 1, n \geq m,$$

$$K_{n,\,m}^k = K_{n-1,\,m}^k + 2kK_{n-m,\,m}^{k-1} + 2K_{n-m,\,m}^k, \quad k \geq 1, n \geq m-1,$$

$$K_{n,\,m}^k = H_{n-1,\,m}^k + 4kH_{n+1-m,\,m}^{k-1} + 4H_{n+1-m,\,m}^k, \quad k \geq 1, n \geq m-1.$$

The sequences $\{H_{n,\,m}^k\}$ and $\{K_{n,\,m}^k\}$ have the following generating functions:

$$\sum_{n=0}^{\infty} H_{n,\,m}^k t^n = \frac{2^k k!\, r^{mk+1}}{(1-t-2t^m)^{k+1}} \quad \text{and} \quad \sum_{n=0}^{\infty} K_{n,\,m}^k t^n = \frac{2^k k!(2-t)t^{mk}}{(1-t-2t^m)^{k+1}}.$$

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