# ELEMENTARY PROBLEMS AND SOLUTIONS

## *Edited by*
## Russ Euler and Jawad Sadek

*Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.*

*If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.*

*Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by February 15, 2001. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".*

**Dedication:** The problems in this issue are dedicated to Dr. Stanley Rabinowitz in recognition of his nine years of devoted service as Editor of the Elementary Problems Section.

### BASIC FORMULAS

The Fibonacci numbers $F_n$ and the Lucas numbers $L_n$ satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

### PROBLEMS PROPOSED IN THIS ISSUE

**B-900** ***Proposed by Richard André-Jeannin, Cosnes et Romain, France***

Show that $\tan(2n\arctan(\alpha))$ is a rational number for every $n \geq 0$.

**B-901** ***Proposed by Richard André-Jeannin, Cosnes et Romain, France***

Let $A_n$ be the sequence defined by $A_0 = 1$, $A_1 = 0$, $A_n = (n-1)(A_{n-1} + A_{n-2})$ for $n \geq 2$. Find

$$\lim_{n\to+\infty} \frac{A_n}{n!}.$$

**B-902** ***Proposed by H.-J. Seiffert, Berlin, Germany***

The Pell polynomials are defined by $P_0(x) = 0$, $P_1(x) = 1$, and $P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x)$ for $n \geq 2$. Show that, for all nonzero real numbers $x$ and all positive integers $n$,

$$\sum_{k=1}^{n} \binom{n}{k} (1-x)^{n-k} P_k(x) = x^{n-1} P_n(1/x).$$

**B-903** ***Proposed by the editor***

Find a closed form for $\sum_{n=0}^{\infty} F_n^2 x^n$.

**B-904** ***Proposed by Richard André-Jeannin, Cosnes et Romain, France***

Find the positive integers $n$ and $m$ such that $F_n = L_m$.

**B-905** ***Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain***

Let $n$ be a positive integer greater than or equal to 2. Determine

$$\frac{(F_n^2+1)F_{n+1}F_{n+2}}{(F_{n+1}-F_n)(F_{n+2}-F_n)}+\frac{F_n(F_{n+1}^2+1)F_{n+2}}{(F_n-F_{n+1})(F_{n+2}-F_{n+1})}+\frac{F_nF_{n+1}(F_{n+2}^2+1)}{(F_n-F_{n+2})(F_{n+1}-F_{n+2})}.$$

## SOLUTIONS

### A Constant Summation

**B-884** ***Proposed by M. N. Deshpande, Aurangabad, India***
***(Vol. 37, no. 4, November 1999)***

Find an integer $k$ such that the expression $F_n^2F_{n+2}^2+kF_{n+1}^2F_{n+2}^2+F_{n+1}^2F_{n+3}^2$ is a constant independent of $n$.

***Composite solution by L. A. G. Dresel, Reading, England, and Maitland A. Rose, University of South Carolina (independently)***

Denoting the given expression by $Q_n$, we have

$$Q_{n+1}-Q_n=k(F_{n+2})^2\{(F_{n+3})^2-(F_{n+1})^2\}+(F_{n+2})^2\{(F_{n+4})^2-(F_n)^2\}.$$

Now

$$(F_{n+3})^2-(F_{n+1})^2=(F_{n+3}-F_{n+1})(F_{n+3}+F_{n+1})=F_{n+2}L_{n+2}$$

and

$$(F_{n+4})^2-(F_n)^2=(F_{n+4}-F_n)(F_{n+4}+F_n)=(F_{n+3}+F_{n+1})(3F_{n+2})=3F_{n+2}L_{n+2},$$

so that $Q_{n+1}-Q_n=(k+3)(F_{n+2})^3L_{n+2}$. Since we require $Q_{n+1}-Q_n=0$ for all $n$, we must have $k=-3$, giving the identity

$$(F_nF_{n+2})^2-3(F_{n+1}F_{n+2})^2+(F_{n+1}F_{n+3})^2=1 \text{ for all } n.$$

*The proposer noted that a similar result holds for the Lucas numbers. The constant k would still be $-3$ but the value of the analogous expression would be 25.*

***Also solved by Gerald Heuer, H.-J. Seiffert, James A. Sellers, Indulis Strazdins, and the proposer.***

### A Unit Summation

**B-885** ***Proposed by A. J. Stam, Winsum, The Netherlands***
***(Vol. 37, no. 4, November 1999)***

For $n>0$, evaluate

$$\sum_{k=0}^{n}(-1)^{n-k}\frac{k}{2n-k}\binom{2n-k}{n}F_{k+1}.$$

***Solution 1 by Kuo-Jye Chen, National Changhua University of Education, Taiwan***

We rewrite the sum as

$$\sum_{k=0}^{n}(-1)^{k}\frac{n-k}{n+k}\binom{n+k}{k}F_{n-k+1}:=A_n,$$

and claim that $A_n = 1$ for $n > 0$.

It is readily seen that

$$A_1 = 1. \tag{1}$$

Next, we show that

$$A_n - A_{n-1} = 0 \quad \text{for } n > 1. \tag{2}$$

Write

$$A_n - A_{n-1} = d_0 + d_1 + d_2 + \cdots + d_{n-1}, \tag{3}$$

where

$$d_k := (-1)^k \left\{ \frac{n-k}{n+k}\binom{n+k}{k} F_{n-k+1} - \frac{n-k-1}{n+k-1}\binom{n+k-1}{k} F_{n-k} \right\}.$$

We compute the following partial sums of (3):

$$d_0 = F_{n-1},$$

$$d_0 + d_1 = -\frac{n-1}{n+1}\binom{n+1}{1} F_{n-2},$$

$$d_0 + d_1 + d_2 = \frac{n-2}{n+2}\binom{n+2}{2} F_{n-3},$$

$$d_0 + d_1 + d_2 + d_3 = -\frac{n-3}{n+3}\binom{n+3}{3} F_{n-4},$$

and, in general,

$$d_0 + d_1 + d_2 + \cdots + d_k = (-1)^k \frac{n-k}{n+k}\binom{n+k}{k} F_{n-k-1}, \quad \text{for } 0 \le k \le n-1. \tag{4}$$

In particular, when $k = n-1$, formula (4) reduces to $d_0 + d_1 + d_2 + \cdots + d_{n-1} = 0$, which completes the proof of (2).

Combining (1) and (2), we obtain, for $n > 0$,

$$\sum_{k=0}^{n} (-1)^{n-k} \frac{k}{2n-k}\binom{2n-k}{n} F_{k+1} = \sum_{k=0}^{n} (-1)^k \frac{n-k}{n+k}\binom{n+k}{k} F_{n-k+1} = 1.$$

***Solution 2 by H.-J. Seiffert, Berlin, Germany***

Define the Fibonacci polynomials and the Lucas polynomials by

$$F_0(x) = 0,\ F_1(x) = 1,\ F_{n+1}(x) = xF_n(x) + F_{n-1}(x),\ n \ge 1,$$

and

$$L_0(x) = 2,\ L_1(x) = x,\ L_{n+1}(x) = xL_n(x) + L_{n-1}(x),\ n \ge 1,$$

respectively. Differentiating the known identity (see [1])

$$\sum_{k=1}^{n} (-1)^{n-k}\binom{2n-1-k}{n-1} x^k L_k(x) = x^{2n}$$

with respect to $x$, using the fact that $L_k'(x) = kF_k(x)$ and $L_k(x) + xF_k(x) = 2F_{k+1}(x)$, and multiplying by $x/2$ gives

$$\sum_{k=1}^{n} (-1)^{n-k}\binom{2n-1-k}{n-1} kx^k F_{k+1}(x) = nx^{2n}.$$

Hence, by $\binom{2n-1-k}{n-1} = \frac{n}{2n-k}\binom{2n-k}{n}$,

$$\sum_{k=0}^{n}(-1)^{n-k}\frac{k}{2n-k}\binom{2n-k}{n}x^k F_{k+1}(x) = x^{2n}.$$

Now, take $x = 1$ to see that the value of the sum in question is 1.

**Reference**

1. R. André-Jeannin & Paul S. Bruckman. "Problem H-479." *The Fibonacci Quarterly* **32.5** (1994):477-78.

***Also solved by Indulis Strazdins and the proposer. One incomplete solution was received.***

## Some Sum

**B-887** ***Proposed by A. J. Stam, Winsum, The Netherlands (Vol. 37, no. 4, November 1999)***

Show that

$$\sum_{k=0}^{n}\binom{y-n-1-k}{n-k}F_{2k+1} = \sum_{k=0}^{n}\binom{y-n-2-k}{n-k}F_{2k+2} = \sum_{j=0}^{n}\binom{y-j}{j}.$$

***Solution by H.-J. Seiffert, Berlin, Germany***

Define the Fibonacci polynomials by $F_0(x) = 0$, $F_1(x) = 1$, and $F_{n+1}(x) = xF_n(x) + F_{n-1}(x)$ for $n \in \mathbb{Z}$. It is known (see [1], identity (60)) that

$$F_{n+1}(x) = \sum_{j=0}^{\lfloor n/2 \rfloor}\binom{n-j}{j}x^{n-2j}, \quad n \in \mathbb{N}_0, \tag{1}$$

and (see [1], identity (39))

$$x^m F_r(x) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}F_{m+r-2j}(x), \quad m \in \mathbb{N}_0, \; r \in \mathbb{Z}. \tag{2}$$

For the nonnegative integer $n$, consider the polynomials

$$P(z) = \sum_{j=0}^{\lfloor n/2 \rfloor}(-1)^j\binom{z}{j}F_{n-2j+1}(x) \quad \text{and} \quad Q(z) = \sum_{j=0}^{\lfloor n/2 \rfloor}\binom{n-z-j}{j}x^{n-2j}.$$

If $z = m$ is an integer such that $0 \le m \le \lfloor n/2 \rfloor$, then by (1) and (2), $P(m) = x^m F_{n-m+1}(x) = Q(m)$. Since $P(z)$ and $Q(z)$ are both polynomials in $z$ of degree not greater than $\lfloor n/2 \rfloor$, we then must have $P(z) = Q(z)$ for all complex numbers $z$. This proves the identity

$$\sum_{j=0}^{\lfloor n/2 \rfloor}(-1)^j\binom{z}{j}F_{n-2j+1}(x) = \sum_{j=0}^{\lfloor n/2 \rfloor}\binom{n-z-j}{j}x^{n-2j}, \tag{3}$$

valid for all $n \in \mathbb{N}_0$ and all complex numbers $x$ and $z$.

To deduce the desired identities, we will use the well-known relation

$$(-1)^j\binom{-a-1}{j} = \binom{a+j}{j}, \quad j \in \mathbb{N}_0, \; a \in \mathbb{C}.$$

Now, using (3) with $n$ replaced by $2n$ and $z = 2n - y$ and then reindexing on the left-hand side $j = n - k$ gives

$$\sum_{k=0}^{n}\binom{y-n-1-k}{n-k}F_{2k+1}(x)=\sum_{j=0}^{n}\binom{y-j}{j}x^{2n-2j}.$$

Similarly, using (3) with $n$ replaced by $2n+1$ and then taking $z=2n+1-y$ yields

$$\sum_{k=0}^{n}\binom{y-n-2-k}{n-k}F_{2k+2}(x)=\sum_{j=0}^{n}\binom{y-j}{j}x^{2n-2j+1}.$$

Finally, take $x=1$.

**Reference**

1. S. Rabinowitz. "Algorithmic Manipulation of Second-Order Linear Recurrences." *The Fibonacci Quarterly* **37.2** (1999):162-77.

***Also solved by the proposer.***

**Determine the Determinant**

**B-888** ***Proposed by A. Arya, J. Fellingham, and D. Schroeder, Ohio State University, OH, and J. Glover, Carnegie Mellon University, PA (Vol. 37, no. 4, November 1999)***

For $n\geq 1$, let $A_n=[a_{i,j}]$ denote the symmetric matrix with $a_{i,i}=i+1$ and $a_{i,j}=\min[i,j]$ for all integers $i$ and $j$ with $i\neq j$.

(a) Find the determinant of $A_n$.

(b) Find the inverse of $A_n$.

***Composite solution to part (a) by L. A. G. Dresel, C. Libis, I. Strazdins, and the proposers.***

Let $D_n$ denote the determinant of $A_n$. Then we have $D_1=2$ and $D_2=5$, and for $n>2$,

$$D_n=\begin{vmatrix}2&\cdot&\cdot&1&1\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&n&n-1\\ 1&\cdot&\cdot&n-1&n+1\end{vmatrix}=\begin{vmatrix}2&\cdot&\cdot&1&1\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&n&n-1\\ 0&0&0&-1&2\end{vmatrix}=\begin{vmatrix}2&\cdot&\cdot&1&0\\ \cdot&\cdot&\cdot&\cdot&0\\ \cdot&\cdot&\cdot&\cdot&0\\ 1&\cdot&\cdot&n&-1\\ 0&0&0&-1&3\end{vmatrix},$$

since the determinant is unchanged if we first subtract the penultimate row from the last row, and then subtract the penultimate column from the last column. Expanding the resulting determinant by its last row, we obtain $D_n=3D_{n-1}-D_{n-2}$. But we have $D_1=2=F_3$ and $D_2=5=F_5$ so that, if we assume that $D_n=F_{2n+1}$ for $n\leq N$, we obtain $D_{N+1}=3D_N-D_{N-1}=3F_{2N+1}-F_{2N-1}=F_{2N+3}$. Hence, by induction, we have $D_n=F_{2n+1}$ for all $n\geq 1$.

*No detailed solution to part (b) was received. The proposers stated that the inverse of $A_n$ is $[b_{ij}]$, where*

$$b_{ii}=\left(\frac{1}{F_{2n+1}}\right)\left(F_{2(n-i)+1}F_{2i-1}+F_{2(n-i)}F_{2i}\right)\ \textit{and}\ b_{ij}=-\left(\frac{1}{F_{2n+1}}\right)\left(F_{2(n-\max\{i,j\})+1}\right)\left(F_{2\min\{i,j\}}\right),\ i\neq j.$$

*However, showing that $[a_{ij}][b_{ij}]=I_n$ involves tedious algebra.*

**Addenda:** We wish to belatedly acknowledge solutions from the following solvers:

Charlie Cook—Problems B-873, B-875, and B-877; Maitland A. Rose—Problem B-878.

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