EVALUATION OF CERTAIN INFINITE SERIES INVOLVING TERMS OF GENERALIZED SEQUENCES

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1. INTRODUCTION AND PRELIMINARIES

This article deals with the *generalized Fibonacci numbers* $U_k(m)$ and the *generalized Lucas numbers* $V_k(m)$ which have already been considered in [9], [11], and elsewhere. These numbers satisfy the second-order recurrence relation

$$W_k(m) = mW_{k-1}(m) + W_{k-2}(m) \quad (k \geq 2) \tag{1.1}$$

where W stands for either U or V, and m is an arbitrary integer. The initial conditions in (1.1) are $W_0(m) = 0$, $W_1(m) = 1$, or $W_0(m) = 2$, $W_1(m) = m$ depending on whether W is U or V. Whenever no misunderstanding can arise, $U_k(m)$ and $V_k(m)$ will be denoted simply by U_k and V_k, respectively.

Closed-form expressions (Binet forms) for these numbers are

$$\begin{cases} U_k(m) = (\alpha_m^k - \beta_m^k) / \Delta_m, \\ V_k(m) = \alpha_m^k + \beta_m^k, \end{cases} \tag{1.2}$$

where

$$\begin{cases} \Delta_m = (m^2+4)^{1/2}, \\ \alpha_m = (m+\Delta_m)/2, \\ \beta_m = (m-\Delta_m)/2, \end{cases} \tag{1.3}$$

so that

$$\alpha_m\beta_m = -1, \;\; \alpha_m + \beta_m = m, \;\; \text{and} \;\; \alpha_m - \beta_m = \Delta_m. \tag{1.4}$$

It can be proved that the extension through negative values of the subscript leads to

$$\begin{cases} U_{-k} = (-1)^{k-1}U_k, \\ V_{-k} = (-1)^k V_k. \end{cases} \tag{1.5}$$

Observe that $U_k(1) = F_k$ and $V_k(1) = L_k$ (the k^{th} Fibonacci and Lucas number, respectively), while $U_k(2) = P_k$ and $V_k(2) = Q_k$ (the k^{th} Pell and Pell-Lucas number, respectively).

The principal aim of this paper is to generalize [14], and some results established in [1], [4], and [12] (see also [5]), by finding an explicit form for the infinite series

$$S_h(r, s, m) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} k^h \frac{r^k}{s^k} V_k, \tag{1.6}$$

where V_k obviously stands for $V_k(m)$ and h, r, and s are positive integers, the last two of which are subject to the restriction,

$$r/s < 1/\alpha_m. \tag{1.7}$$

By using the Binet form (1.2) for V_k, it can be proved readily (e.g., see [6], pp. 266-67]) that the inequality (1.7) is a necessary and sufficient condition for the sum (1.6) to converge.

The paper is set out as follows. An explicit form for the sum (1.6) (the main result) is established in Section 2, while special cases of it are considered in Section 3. A glimpse of some possible extensions is caught in Sections 4 and 5.

2. THE MAIN RESULT

The main result established in this paper reads as follows.

Theorem 1:

$$S_h(r, s, m) = \frac{\sum_{i=0}^{h+1}\sum_{j=1}^{h}\binom{h+1}{i}A_{h,j}s^{i+j}r^{2h-i-j+2}V_{i-j}}{(s^2 - srm - r^2)^{h+1}}, \tag{2.1}$$

where

$$A_{h,j} = \sum_{n=0}^{j}(-1)^n\binom{h+1}{n}(j-n)^h \tag{2.2}$$

are the *Eulerian numbers* (e.g., see [3]), and the inequality (1.7) is assume to be satisfied. We recall that the numbers $A_{h,j}$ may be expressed equivalently [2] as

$$A_{h,j} = (-1)^j\sum_{n=1}^{j}(-1)^n\binom{h+1}{j-n}n^h. \tag{2.2'}$$

Observe that (2.1) involves the use of (1.5). For illustration only, we show the first few numbers $A_{h,j}$ ($A_{h,j} \neq 0$ for $1 \leq j \leq h$; $A_{h,j} = A_{h,h-j+1}$):

$$\begin{aligned}
&A_{1,1} = 1,\\
&A_{2,1} = A_{2,2} = 1,\\
&A_{3,1} = A_{3,3} = 1;\ A_{3,2} = 4,\\
&A_{4,1} = A_{4,4} = 1;\ A_{4,2} = A_{4,3} = 11,\\
&A_{5,1} = A_{5,5} = 1;\ A_{5,2} = A_{5,4} = 26;\ A_{5,3} = 66,\\
&A_{6,1} = A_{6,6} = 1;\ A_{6,2} = A_{6,5} = 57;\ A_{6,3} = A_{6,4} = 302.
\end{aligned}$$

Proof of Theorem 1: First, recall (e.g., see [10]) that

$$\sum_{k=1}^{\infty}k^h y^k = \frac{1}{(1-y)^{h+1}}\sum_{j=1}^{h}A_{h,j}y^{h-j+1} \quad (|y| < 1). \tag{2.3}$$

Then use (1.2), (2.3), and (1.4) along with (1.6) to write

$$\begin{aligned}
S_h(r, s, m) &= \sum_{k=1}^{\infty}k^h(r\alpha_m/s)^k + \sum_{k=1}^{\infty}k^h(r\beta_m/s)^k\\
&= \frac{\sum_{j=1}^{h}A_{h,j}(r\alpha_m/s)^{h-j+1}}{(1-r\alpha_m/s)^{h+1}} + \frac{\sum_{j=1}^{h}A_{h,j}(r\beta_m/s)^{h-j+1}}{(1-r\beta_m/s)^{h+1}}
\end{aligned}$$

$$= \frac{(r\alpha_m)^{h+1}\sum_{j=1}^{h} A_{h,j}(r\alpha_m/s)^{-j}}{(s-r\alpha_m)^{h+1}} + \frac{(r\beta_m)^{h+1}\sum_{j=1}^{h} A_{h,j}(r\beta_m/s)^{-j}}{(s-r\beta_m)^{h+1}}$$

$$= \frac{(sr\alpha_m+r^2)^{h+1}\sum_{j=1}^{h} A_{h,j}(r\alpha_m/s)^{-j} + (sr\beta_m+r^2)^{h+1}\sum_{j=1}^{h} A_{h,j}(r\beta_m/s)^{-j}}{(s^2-srm-r^2)^{h+1}}. \tag{2.4}$$

By using the binomial expansions of $(sr\alpha_m+r^2)^{h+1}$ and $(sr\beta_m+r^2)^{h+1}$, the equality (2.4) becomes

$$S_h(r,s,m) = \left[\sum_{i=0}^{h+1}\sum_{j=1}^{h}\binom{h+1}{i}A_{h,j}s^{i+j}r^{2h+2-i-j}\alpha_m^{i-j} + \sum_{i=0}^{h+1}\sum_{j=1}^{h}\binom{h+1}{i}A_{h,j}s^{i+j}r^{2h+2-i-j}\beta_m^{i-j}\right] / (s^2-srm-r^2)^{h+1},$$

whence, by using the Binet form (1.4), one gets the right-hand side of (2.1). Q.E.D.

3. SPECIAL CASES

In this section we consider ordered pairs (r, s) subject to (1.7) for which $S_h(r,s,m)$, beyond being a positive *integer*, has a form that is much more compact than (2.1). First, we need the following two propositions.

Proposition 1:

$$U_{2n+1}^2 - mU_{2n+1}U_{2n} - U_{2n}^2 = 1. \tag{3.1}$$

Proposition 2: If we let δ stand for either α or β, then

$$\delta_m U_{2n+1} + U_{2n} = \delta_m^{2n+1}. \tag{3.2}$$

Proof of Proposition 1: Rewrite the left-hand side of (3.1) as

$$U_{2n+1}^2 - U_{2n}(mU_{2n+1}+U_{2n})$$
$$= U_{2n+1}^2 - U_{2n}U_{2n+2} \quad \text{[by (1.1)]}$$
$$= -(-1)^{2n+1} = 1 \quad \{\text{by the } \textit{Simson formula} \text{ (e.g., see (2.18) of [9])}\}.$$ Q.E.D.

Proof of Proposition 2: For the sake of brevity, we shall prove only the case $\delta = \alpha$. Using (1.2), rewrite the left-hand side of (3.2) as

$$[(\alpha_m^{2n+1}-\beta_m^{2n+1})\alpha_m + \alpha_m^{2n} - \beta_m^{2n}]/\Delta_m$$
$$= (\alpha_m^{2n+2}+\alpha_m^{2n})/\Delta_m \quad \text{[by (1.4)]}$$
$$= \alpha^{2n+1}(\alpha_m+\alpha_m^{-1})/\Delta_m$$
$$= \alpha^{2n+1}(\alpha_m-\beta_m)/\Delta_m \quad \text{[by (1.4)]}$$
$$= \alpha^{2n+1}U_1(m) = \alpha^{2n+1}\cdot 1 = \alpha^{2n+1}.$$ Q.E.D.

Now, after observing that $U_{2n}/U_{2n+1} < 1/\alpha_m$ for all m, we state the following theorem.

Theorem 2:

$$S_h(U_{2n}, U_{2n+1}, m) = \sum_{j=1}^{h} A_{h,j} U_{2n+1}^j U_{2n}^{h+1-j} V_{(2n+1)(h+1)-j}. \tag{3.3}$$

Proof: Replace r by U_{2n} and s by U_{2n+1} in (2.4), and use Propositions 1 and 2 to write

$$\begin{aligned} S_h(U_{2n}, U_{2n+1}, m) &= (U_{2n}\alpha_m^{2n+1})^{h+1} \sum_{j=1}^{h} A_{h,j} \left(\frac{U_{2n}}{U_{2n+1}} \alpha_m \right)^{-j} \\ &\quad + (U_{2n}\beta_m^{2n+1})^{h+1} \sum_{j=1}^{h} A_{h,j} \left(\frac{U_{2n}}{U_{2n+1}} \beta_m \right)^{-j} \\ &= \sum_{j=1}^{h} A_{h,j} U_{2n+1}^j U_{2n}^{h+1-j} \left[\alpha_m^{(2n+1)(h+1)-j} + \beta_m^{(2n+1)(h+1)-j} \right]. \end{aligned}$$

By using the Binet form (1.2), the right-hand side of (3.3) is immediately obtained. Observe that, by solving in integers the Pell equation (1) on page 100 in [13], it can be proved that the pairs (U_{2n}, U_{2n+1}) are the *only* pairs (r, s) for which the denominator of (2.1) equals 1. Q.E.D.

A very special case $(n = m = 1)$ of (3.3) is

$$S_h(1, 2, 1) = \sum_{k=1}^{\infty} k^h L_k / 2^k = \sum_{j=1}^{h} A_{h,j} 2^j L_{3(h+1)-j}. \tag{3.4}$$

The proof of the identity

$$S_h(V_{2n-1}, V_{2n}, m) = \left[\sum_{j=1}^{h} A_{h,j} V_{2n}^j V_{2n-1}^{h+1-j} V_{2n(h+1)-j} \right] / \Delta_m^2, \tag{3.5}$$

which is the Lucas analog of (3.3), is left as an exercise for the interested reader.

4. EXTENSIONS

It is obvious that the result (2.1) allows us to evaluate the more general series

$$\sum_{k=1}^{\infty} p(k) \frac{r^k}{s^k} V_k, \tag{4.1}$$

where $p(k)$ is a polynomial in k. As a minor example, we offer the following identity:

$$S(n, m) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} (k^2 + k) \frac{U_{2n}^k}{U_{2n+1}^k} V_k = 2U_{2n} U_{2n+1}^2 V_{6n+1}. \tag{4.2}$$

Proof: By (3.3), write

$$\begin{aligned} S(n, m) &= S_1(U_{2n}, U_{2n+1}, m) + S_2(U_{2n}, U_{2n+1}, m) \\ &= A_{1,1} U_{2n+1} U_{2n} V_{2(2n+1)-1} + A_{2,1} U_{2n+1} U_{2n}^2 V_{3(2n+1)-1} + A_{2,2} U_{2n+1}^2 U_{2n} V_{3(2n+1)-2}. \end{aligned}$$

Recalling that $A_{1,1} = A_{2,1} = A_{2,2} = 1$,

$$S(n, m) = U_{2n} U_{2n+1} (V_{4n+1} + U_{2n} V_{6n+2} + U_{2n+1} V_{6n+1}). \tag{4.3}$$

After some tedious manipulations involving the use of (1.2), it can be proved that

$$V_{4n+1}+U_{2n}V_{6n+2}=U_{2n+1}V_{6n+1}. \tag{4.4}$$

The identity (4.2) readily follows from (4.3) and (4.4). Q.E.D.

We also tried to extend (1.6) to negative values of h. For $h=-1$, we get

$$\sum_{k=1}^{\infty}\frac{1}{k}\frac{r^kV_k}{s^k}=-\ln\frac{s^2-rsm-r^2}{s^2}\quad (r/s<1/\alpha_m). \tag{4.5}$$

Proof: By using (1.2), (1.4), and the identity 1.513.4 on page 44 in [7],

$$\sum_{k=1}^{\infty}k^{-1}y^k=-\ln(1-y)\quad (|y|<1), \tag{4.6}$$

the left-hand side of (4.5) can be rewritten as

$$-\ln(1-r\alpha_m/s)-\ln(1-r\beta_m/s)=-\ln[(1-r\alpha_m/s)(1-r\beta_m/s)]$$
$$=-\ln\frac{s^2-rsm-r^2}{s^2}.\quad \text{Q.E.D.}$$

Special cases of (4.5) are

$$\sum_{k=1}^{\infty}\frac{1}{k}\frac{U_{2n}^kV_k}{U_{2n+1}^k}=-\ln\frac{1}{U_{2n+1}^2}=2\ln U_{2n+1}. \tag{4.7}$$

For $h=-2$, we should have at our disposal a closed-form expression for $\sum_{k=1}^{\infty}k^{-2}y^k$. By (4.6), it can readily be seen that

$$\sum_{k=1}^{\infty}k^{-2}y^k=-\int\frac{\ln(1-y)}{y}dy\quad (|y|<1). \tag{4.8}$$

Unfortunately, the right-hand side of (4.8) cannot be expressed in terms of elementary transcendental functions.

We conclude this paper by establishing the following identity:

$$\sum_{k=1}^{\infty}\frac{1}{k^2+k}\frac{U_{2n}^kV_k}{U_{2n+1}^k}=2+\frac{U_{2n+1}}{U_{2n}}(m\ln U_{2n+1}-2n\Delta_m\ln\alpha_m)+2\ln U_{2n+1}. \tag{4.9}$$

Proof: By using (1.2) and the identity 1.513.5 on page 45 of [7],

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)}y^k=1-\frac{1-y}{y}\ln\frac{1}{1-y}\quad (|y|<1), \tag{4.10}$$

first write

$$X(r,s)\stackrel{\text{def}}{=}\sum_{k=1}^{\infty}\frac{1}{k^2+k}\frac{r^kV_k}{s^k}=1-\frac{1-r\alpha_m/s}{r\alpha_m/s}\ln\frac{1}{1-r\alpha_m/s}+1-\frac{1-r\beta_m/s}{r\beta_m/s}\ln\frac{1}{1-r\beta_m/s}$$
$$=2-\left[\frac{s-r\alpha_m}{r\alpha_m}\ln\frac{s}{s-r\alpha_m}+\frac{s-r\beta_m}{r\beta_m}\ln\frac{s}{s-r\beta_m}\right]$$
$$=2-\left[\frac{s}{r\alpha_m}\ln\frac{s}{s-r\alpha_m}+\frac{s}{r\beta_m}\ln\frac{s}{s-r\beta_m}\right]+\ln\frac{s^2}{(s-r\alpha_m)(s-r\beta_m)}.$$

After some manipulations involving the use of (1.4), one obtains

$$X(r,s)=2+\frac{sm}{r}\ln s+\ln\frac{s^2}{s^2-rsm-r^2}+\frac{s}{r\alpha_m}\ln(s-r\alpha_m)+\frac{s}{r\beta_m}\ln(s-r\beta_m). \tag{4.11}$$

Then, replace r by U_{2n} and s by U_{2n+1} in (4.11), thus getting

$$\begin{aligned}X(U_{2n},U_{2n+1})&=2+\frac{mU_{2n+1}}{U_{2n}}+(2\ln U_{2n+1}-\ln 1)\\&+\left[\frac{U_{2n+1}}{\alpha_mU_{2n}}\ln(U_{2n+1}-\alpha_mU_{2n})+\frac{U_{2n+1}}{\beta_mU_{2n}}\ln(U_{2n+1}-\beta_mU_{2n})\right].\end{aligned} \tag{4.12}$$

Using (1.2) and (1.4), the expression within square brackets becomes

$$\begin{aligned}\left[\frac{U_{2n+1}}{\alpha_mU_{2n}}\ln\beta_m^{2n}+\frac{U_{2n+1}}{\beta_mU_{2n}}\ln\alpha_m^{2n}\right]&=\frac{U_{2n+1}}{U_{2n}}\left[\frac{1}{\alpha_m}\ln\frac{1}{\alpha_m^{2n}}+\frac{1}{\beta_m}\ln\alpha_m^{2n}\right]\\&=\frac{U_{2n+1}}{U_{2n}}(\beta_m^{-1}-\alpha_m^{-1})\ln\alpha_m^{2n}=-\frac{U_{2n+1}}{U_{2n}}2n\Delta_m\ln\alpha_m.\end{aligned} \tag{4.13}$$

The right-hand side of (4.9) readily follows from (4.12) and (4.13). Q.E.D.

5. CONCLUDING COMMENTS

For the sake of brevity, we confined ourselves to considering series involving only the numbers $V_k(m)$. On the other hand, analogous results for $U_k(m)$ can readily be obtained by paralleling the arguments in Sections 2, 3, and 4. This can be done as an exercise by the interested reader.

The investigation of infinite series involving terms of the more general sequences $\{W_k(a,b;m,q)\}_{k=0}^{\infty}$ (see [8]) seems to be a substantial extension of our study, and will be the object of a future work. For $a=0$ and $b=-q=1$, this investigation might lead to an interesting generalization of Theorem 10 of [11].

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IN MEMORIAM—LEONARD CARLITZ

Leonard Carlitz, a long-time friend and supporter of The Fibonacci Association, passed away on September 17, 1999. For many years Carlitz was on the editorial board of *The Fibonacci Quarterly*, and between 1963 and 1984 he published 72 articles in the *Quarterly* (including 19 joint papers and 7 short notes).

Carlitz was born in 1907 in Philadelphia, and he grew up in that city. He won a scholarship to the University of Pennsylvania where he completed his AB degree in 1927, his MA degree in 1928, and his Ph.D. in 1930—all in mathematics. His Ph.D. thesis advisor was H. H. Mitchell, who had been a student of Oswald Veblen who, in turn, had studied under E. H. Moore. Inspired by earlier research of Emil Artin, Carlitz wrote his dissertation of "Galois Fields of Certain Types." This work appeared under the same title in the 1930 *Transactions of the AMS* (Vol. 32, pp. 451-472).

Carlitz spent the 1930-1931 academic year as a National Research Council Fellow studying with E. T. Bell at the California Institute of Technology, and he spent the 1931-1932 academic year with G. H. Hardy in Cambridge, England, as an International Research Fellow. He taught at Duke University, where he was James B. Duke Professor of Mathematics, from 1932 until his retirement in 1977. At Duke he was research advisor to 44 Ph.D. students and 51 MS students. He was also involved in the early planning for the *Duke Mathematical Journal* (established 1935), and he served for many years as the managing editor. He spent the year 1935-1936 at the Institute for Advanced Study.

In the summer of 1931, between Caltech and Cambridge, Carlitz met and married Clara Skaler. They had two children: Michael (born 1939) and Robert (born 1945). Mrs. Carlitz died in 1990.

Carlitz was a prolific and insightful researcher, with 771 publications in many different areas of mathematics. He will be remembered as a first-class mathematician, an inspiring teacher, and a kind, generous man. More information about him, including some personal anecdotes, can be found in the excellent tribute by Joel Brawley: "Dedicated to Leonard Carlitz: The Man and His Work" [*Finite Fields and Their Applications* **1** (1995):135-151].

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