FIBONACCI NUMBERS, GENERATING SETS AND HEXAGONAL PROPERTIES

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(Submitted September 1998-Final Revision September 1999)

1. INTRODUCTION

It is well known that the entries $p(n, t) = \binom{n}{t}$, $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$, $0 \le t \le n$, of Pascal's triangle satisfy *equal product* and *equal gcd* (*greatest common divisor*) *hexagonal properties*: the two alternate triads arising from the six binomial coefficients surrounding any given entry in Pascal's array have equal product and equal gcd[7, 8, 10, 14, 15] (see Fig. 1). Pascal's array fails to satisfy an *equal lcm* (*least common multiple*) *hexagonal property*.

1
1 1
1 2 1
1 3 3 1
1 **4** **6** 4 1
1 **5** 10 **10** 5 1
1 6 **15** **20** 15 6 1

FIGURE 1. A Typical Hexagon in Pascal's Array
Notice: $5 \cdot 6 \cdot 20 = 4 \cdot 10 \cdot 15$ and $\gcd(5, 6, 20) = 1 = \gcd(4, 10, 15)$.

These observations of the early 1970s initiated the search for and discovery of many beautiful configurations within Pascal's array satisfying equal product, equal gcd and even equal lcm properties [2, 4, 5, 12]. Similar properties have been discovered for other arrays, such as the Binomial triangle, and for higher-dimensional "pyramids" of multinomials, multi-Fibonomials, and the like [1, 3, 9, 11, 13].

Recently, R. P. Grimaldi [6] discovered hexagonal properties occurring within the array with entries $g(n, t) = F_t F_{n+1-t}$ ($n \in \mathbb{N}, 1 \le t \le n$). This array arose from a study of *generating sets*, that is, subsets S of $[n] = \{1, 2, \ldots, n\}$ satisfying $S \cup (S+1) = [n+1]$, where $S+1 = \{s+1 : s \in S\} \subseteq [n+1]$. Counting the number of such sets that contain the particular element t produces the quantity $g(n, t)$. It is a surprise to learn that this array satisfies both the equal product and equal gcd hexagonal properties.

In this paper we study higher-order generating sets, that is, subsets S of $[n]$ satisfying $S \cup (S+k) = [n+k]$ for some $k \in \mathbb{N}$, $1 \le k \le n$ (where $S = k = \{s+k : s \in S\}$). We call such a set a k^{th}*-order generating set* and say that S generates $[n+k]$.

Setting $g^k(n, t)$ to be the number of such sets that contain the particular element $t \in \mathbb{N}$, we are thus given, for each $k \in \mathbb{N}$, new arrays with potential hexagonal properties. We will show that the entries $g^k(n, t)$ are products of $k+1$ Fibonacci numbers and explore the extent to which equal product and equal gcd hexagonal properties hold.

In this paper it is convenient to set $F_n = 0$ for n a nonpositive integer. For $x \in \mathbb{R}$, we use $\lceil x \rceil$ to denote the least integer greater than or equal to x and $\lfloor x \rfloor$ the greatest integer less than or equal to x.

We begin by recalling some standard properties of the Fibonacci sequence that will be used throughout this work.

2. THE FIBONACCI NUMBERS

An easy inductive argument establishes:

A. $\gcd(F_t, F_{t+1}) = 1$ for all $t > 0$.

Since $\gcd(F_t, F_{t+2}) = \gcd(F_t, F_{t+1} + F_t) = \gcd(F_t, F_{t+1})$, we have:

B. $\gcd(F_t, F_{t+2}) = 1$ for all $t > 0$.

We also have the key relation:

C. $F_{t+r} = F_r F_{t+1} + F_{r-1} F_t$ for all $t \geq 0, r \geq 1$.

This is easily established by an induction argument on r. With **A** and **B** it has the following consequences:

D. Let $r, t \geq 0, d \in \mathbb{N}$. If $d | F_t$ and $d | F_{t+r}$, then $d | F_r$. Consequently $\gcd(F_t, F_{t+r}) | F_r$.

E. Let $r, t \geq 0, d \in \mathbb{N}$. Suppose $d | F_t$ and $d | F_{t+r}$. If, for $k \in \mathbb{N}$, $d | F_k$, then $d | F_{k \pm r}$.

[By **D**, $d | F_r$. **A** and **C** now show $d | F_{k+r}$. If $r \leq k$, then; $F_k = F_r F_{k+1-r} + F_{r-1} F_{k-r}$, from which it follows that $d | F_{k-r}$. The result is trivial for $r > k$.]

F. Let $r, k \geq 0, d \in \mathbb{N}$. If $d | F_r$ and $d | F_k$, then $d | F_{k \pm mr}$ for any $m \in \mathbb{N}$.

[This follows from repeated application of **E** with $t = 0$.]

G. $F_k | F_{mk}$ for $m, k \in \mathbb{N}$.

[Set $r = k$ in **F**.]

H. Let $d \in \mathbb{N}$ and let F_a be the first Fibonacci number ($a \in \mathbb{N}$) so that $d | F_a$. Let $k \geq 0$. Then $d | F_k \Leftrightarrow a | k$.

[($\Leftarrow$) follows from **G**. For ($\Rightarrow$), write $k = ma + b$ with $0 \leq b < a$, $m \in \mathbb{N}$. **F** shows $d | F_b$, a contradiction unless $b = 0$.]

I. For $r, k \in \mathbb{N}$, $\gcd(F_r, F_k) = F_{\gcd(r,k)}$.

[That $F_{\gcd(r,k)}$ is a common divisor of F_r and F_k follows from **G**, **F**, and the Euclidean algorithm show that $\gcd(F_r, F_k) = F_{\gcd(r,k)}$.]

3. A CONSTRUCTIVE MODEL

We have the following familiar model for constructing Fibonacci numbers: For $n, k, t \in \mathbb{N}$, let

S_n = the set of all n bit sequences beginning and ending with 1 and containing no two consecutive 0's.

S_n^k = the subset of those sequences that contain precisely k 1's.

$S_n(t)$ = the subset of all those sequences containing a 1 in the t^{th} place.

We have:

J. $|S_n| = F_n$ for all $n \in \mathbb{N}$.

Proof: Clearly $|S_1| = |S_2| = 1$ and, by considering the choice of the penultimate term in an n bit sequence, we see that $|S_n| = |S_{n-1}| + |S_{n-2}|$ for $n \geq 3$. □

K. $|S_n^k| = \binom{k-1}{n-k}$ provided $n-k \leq k-1$. (It is zero otherwise.)

Proof: There are $n-k$ zeros "to be placed" in the $k-1$ spaces between the ones. □

This yields:

L. $F_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k}$.

Proof: As S_n is the disjoint union of the subsets S_n^k, $0 \leq k \leq n$, $|S_n| = \Sigma_{k=0}^{n} |S_n^k|$. □

M. $|S_n(t)| = F_t F_{n+1-t}$.

Proof: Clearly $S_n(t)$ is isomorphic as a set to $S_t \times S_{n+1-t}$. □

Later, we will denote $|S_n(t)|$ by $g^1(n, t)$. Observe that, by reversing the strings, we have the symmetry, $g^1(n, t) = g^1(n, n+1-t)$. This symmetry appears in many of the tables presented later in this paper.

4. GENERATING SETS

An n bit sequence determines a subset $S \subseteq [n]$ and vice versa (declare $t \in S$ if and only if the t^{th} place of that sequence is a one). Subsets arising from binary sequences as described in the previous section are the generating sets (of order 1). By **J** there are F_n generating subsets of $[n]$ of order 1.

More generally, let $h_k(n)$ be the number of subsets of $[n]$ of order k that generate $[n+k]$. A table for $h_k(n)$ appears in Figure 2.

$h_k(n)$	k=1	2	3	4	5	6	7	8	9	10
n=1	1									
2	1	1								
3	2	1	1							
4	3	1	1	1						
5	5	2	1	1	1					
6	8	4	1	1	1	1				
7	13	6	2	1	1	1	1			
8	21	9	4	1	1	1	1	1		
9	34	15	8	2	1	1	1	1	1	
10	55	25	12	4	1	1	1	1	1	1

FIGURE 2

Theorem 4.1: For $n, k \in \mathbb{N}$, $h_k(n)$ is a product of k Fibonacci numbers. Precisely

$$h_k(n) = \left(F_{\lceil \frac{n}{k} \rceil}\right)^r \left(F_{\lfloor \frac{n}{k} \rfloor}\right)^{k-r},$$

where $n \equiv r \pmod{k}$ with $1 \le r \le k$.

Proof: Clearly $h_k(n) = 0$ if $n < k$ and the theorem is true. Assume then that $n \ge k$. Write $n = mk + r$ with $1 \le r \le k$. (Here, $m = \lceil \frac{n}{k} \rceil - 1$.)

Let $H_k(n)$ be the set of all n bit sequences that correspond to a generating set of order k. Any such sequence must contain a 1 in the first and last k places. It also has the property that if the t^{th} place fails to be a 1 then the $t-k^{\text{th}}$ place must be. Thus, we can partition the sequence into k intertwined subsequences corresponding to those place numbers congruent mod k. Each such subsequence corresponds to a first-order generating set. Counting the lengths of these subsequences, we see that we have a set isomorphism

$$H_k(n) = \underbrace{S_{m+1} \times \cdots \times S_{m+1}}_{r} \times \underbrace{S_m \times \cdots \times S_m}_{k-r}.$$

The theorem follows from **J**. □

5. ARRAYS

For $n, t, k \in \mathbb{N}$, let $g^k(n, t)$ be the number of generating subsets of $[n]$ of order k that contain t. Figures 3, 4, and 5 give tables for $g^k(-,-)$ for $k = 1, 2$, and 3, respectively.

$g^1(n,t)$ (rows n, columns t)	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	1								
3	2	1	2							
4	3	2	2	3						
5	5	3	**4**	**3**	5					
6	8	5	**6**	6	**5**	8				
7	13	8	10	**9**	**10**	8	13			
8	21	13	16	15	15	16	13	21		
9	34	21	26	24	25	24	26	21	34	
10	55	34	42	39	40	40	39	42	34	55

FIGURE 3

$g^2(n,t)$	1	2	3	4	5	6	7	8	9	10
1	0									
2	1	1								
3	1	1	1							
4	1	1	1	1						
5	2	2	1	2	2					
6	4	4	2	2	4	4				
7	6	6	4	2	4	6	6			
8	9	9	6	6	6	6	9	9		
9	15	15	9	10	12	10	9	15	15	
10	25	25	15	15	20	20	15	15	25	25

(rows: n; columns: t)

FIGURE 4

$g^3(n,t)$	1	2	3	4	5	6	7	8	9	10
1	0									
2	0	0								
3	1	1	1							
4	1	1	1	1						
5	1	1	1	1	1					
6	1	1	1	1	1	1				
7	2	2	2	1	2	2	2			
8	4	4	4	2	2	4	4	4		
9	8	8	8	4	4	4	8	8	8	
10	11	11	11	7	5	5	7	11	11	11

(rows: n; columns: t)

FIGURE 5

Observation **M** establishes $g^1(n,t)=F_tF_{n+1-t}$ (see also [6]). We now determine the general formula for $g^k(n,t)$.

Theorem 5.1: For $n,k,t\in\mathbb{N}$, if $n\equiv r \pmod k$ and $t\equiv i \pmod k$ with $1\le r.\ i\le k$, then

$$g^k(n,t)=\begin{cases}\left(F_{\lceil\frac{n}{k}\rceil}\right)^{r-1}\left(F_{\lfloor\frac{n}{k}\rfloor}\right)^{k-r}F_{\lceil\frac{t}{k}\rceil}F_{\lceil\frac{n}{k}\rceil-\lceil\frac{t}{k}\rceil+1} & \text{if } i\le r,\\ \left(F_{\lceil\frac{n}{k}\rceil}\right)^{r}\left(F_{\lfloor\frac{n}{k}\rfloor}\right)^{k-r-1}F_{\lceil\frac{t}{k}\rceil}F_{\lfloor\frac{n}{k}\rfloor-\lceil\frac{t}{k}\rceil+1} & \text{if } i>r.\end{cases}$$

Proof: Let $H_k(n,t)$ be the set of n bit sequences arising from k^{th}-order generating sets $S\subseteq[n]$ containing the element t. Thus, we are declaring that a 1 must always appear in the t^{th} place. This 1 occurs in the $\lceil\frac{t}{k}\rceil^{\text{th}}$ place of the i^{th} subsequence corresponding to the i^{th} congruent class of place numbers mod k. Writing $n=mk+r$, we have a set isomophism

$$H_k(n,t)\cong S_{m+1}\times\cdots\times \overset{i^{\text{th}}\text{ place}}{S_{m+1}(\lceil\tfrac{t}{k}\rceil)}\times\cdots\times S_m$$

if $i>r$. The result follows. □

6. EQUAL PRODUCT HEXAGONAL PROPERTY

In [6], R. P. Grimaldi observed and proved that the $g^1(-,-)$ array (Fig. 3) satisfies the equal product hexagonal property. For example, the two alternate triads in the six entries 6, 4, 3, 5, 10, 9 surrounding $g^1(6,4)=6$ in a (skewed) hexagon have equal products: $6\cdot3\cdot10=4\cdot5\cdot9$.

We call such a hexagon a *hexagon of radius 1*. It is *centered* about $g^1(6,4)$. We observe here that Figure 3 satisfies the equal product property for all hexagons of greater radii. For example, taking two steps outward from the same center $g^1(6,4)$ in the direction of the vertices of the original (skewed) hexagon yields six entries 5, 2, 3, 8, 16, 15 whose alternate triads again satisfy $5\cdot3\cdot16=2\cdot8\cdot15$. We call such a configuration of six entries a *hexagon of radius 2*. The notion of a hexagon of general radius r is defined similarly.

Although all hexagons of arbitrary radius in Figure 3 satisfy the equal product property, the same is not true for the arrays in Figures 4 and 5, or in a general array $g^k(-,-)$, $k\ge2$. Only those hexagons with radius divisible by k can be guaranteed to satisfy the equal product property.

Theorem 6.1 (Equal Product Hexagonal Property): For $n,k,t,a\in\mathbb{N}$,

$$g^k(n-ak,t-ak)\cdot g^k(n,t+ak)\cdot g^k(n+ak,t)=g^k(n-ak,t)\cdot g^k(n,t-ak)\cdot g^k(n+ak,t+ak).$$

Proof: Suppose $n\equiv r \pmod k$, $t\equiv i \pmod k$ with $1\le i,\ r\le k$. Then

$$n-ak,n,n+ak\equiv r \pmod k,$$
$$t-ak,a,t+ak\equiv i \pmod k.$$

Assume $i\le r$. Then, by Theorem 5.1,

$$g^k(n\pm ak,t)=\left(F_{\lceil\frac{n}{k}\rceil\pm a}\right)^{r-1}\left(F_{\lfloor\frac{n}{k}\rfloor\pm a}\right)^{k-r}F_{\lceil\frac{t}{k}\rceil}F_{\lceil\frac{n}{k}\rceil-\lceil\frac{t}{k}\rceil+1\pm a},$$

$$g^k(n,t\pm ak)=\left(F_{\lceil\frac{n}{k}\rceil}\right)^{r-1}\left(F_{\lfloor\frac{n}{k}\rfloor}\right)^{k-r}F_{\lceil\frac{t}{k}\rceil\pm a}F_{\lceil\frac{n}{k}\rceil-\lceil\frac{t}{k}\rceil+1\mp a},$$

$$g^k(n \pm ak, t \pm ak) = \left(F_{\lceil \frac{n}{k} \rceil \pm a}\right)^{r-1} \left(F_{\lfloor \frac{n}{k} \rfloor \pm a}\right)^{k-r} F_{\lceil \frac{t}{k} \rceil \pm a} F_{\lceil \frac{n}{k} \rceil - \lceil \frac{t}{k} \rceil + 1},$$

and the result is easy to establish. The case $i > r$ is proved similarly. □

7. EQUAL gcd HEXAGONAL PROPERTY

7.1 The $g^1(-,-)$ Array

As established by Grimaldi [6], $g^1(-,-)$ also satisfies the equal gcd hexagonal property for radius 1 hexagons. Inspection of Figure 3 might encourage one to suspect that the equal gcd property also holds for higher sized hexagons but this turns out to be false. Consider the hexagon of radius 2 centered in row $n = 23$ about $g^1(23, 10)$. Here the greatest common divisors of the alternate triads are:

$$\gcd(g^1(21,10), g^1(23,8), g^1(25,12)) = \gcd(15 \cdot 144, 21 \cdot 987, 144 \cdot 377) = 9,$$
$$\gcd(g^1(21,8), g^1(25,10), g^1(23,12)) = \gcd(21 \cdot 377, 55 \cdot 987, 144 \cdot 144) = 3.$$

(Incidentally, this is the first instance of failed equal gcd for this array.) Note, however, that both gcd's are composed of powers of the same prime. We will say an array satisfies a *weak gcd hexagonal property* for hexagons of radius r if the greatest common divisors of the alternate triads in any hexagon of that radius are composed of positive powers of the same primes.

Lemma 7.1: The array $g^1(-,-)$ satisfies the weak gcd hexagonal property for all hexagons of arbitrary size. That is, for $n, t, r \in \mathbb{N}$,

$$\gcd(g^1(n-r, t-r), g^1(n, t+r), g^1(n+r, t))$$

and

$$\gcd(g^1(n-r, t), g^1(n, t-r), g^1(n+r, t+r))$$

are composed of positive powers of the same primes.

Proof: By observation **M**,

$$g^1(n-r, t-r) = F_{t-r} F_{n+1-t},$$
$$g^1(n, t+r) = F_{t+r} F_{n+1-t-r},$$
$$g^1(n+r, t) = F_t F_{n+1-t+r},$$

and

$$g^1(n-r, t) = F_t F_{n+1-t-r},$$
$$g^1(n, t-r) = F_{t-r} F_{n+1-t-r},$$
$$g^1(n+r, t+r) = F_{t+r} F_{n+1-t}.$$

Suppose p, a prime, is a common divisor of the first triad. (The case where p is a common divisor of the second triad is proved similarly.) We have four possibilities:

i) p divides each of F_{t-r}, F_{t+r}, and F_t.
ii) p divides each of $F_{n+1-t-r}$, $F_{n+1-t+r}$, and F_{n+1-t}.
iii) p divides two of F_{t-r}, F_{t+r}, and F_t but not the third.
iv) p divides two of $F_{n+1-t-r}$, $F_{n+1-t+r}$, and F_{n+1-t} but not the third.

It is clear that cases i) and ii) imply that p is a common divisor of the second triad.

Consider case iii). By observation, **E**, it must be the case that $p|F_{t-r}$ and $p|F_{t+r}$ but $p\nmid F_t$. By **D**, $p|F_{2r}$. Since $p|g^1(n+r,t)$, we have $p|F_{n+1-t+r}$. Consequently, by **E**, $p|F_{n+1-t-r}$ and p is a common divisor of the second triad.

Case iv) is established similarly. □

We can now quickly establish Grimaldi's result.

Corollary 7.2: All alternate triads for radius 1 hexagons in the $g^1(-,-)$ array have greatest common divisor equal to 1. Consequently, the equal gcd hexagonal property holds for such hexagons.

Proof: We see from the proof of Lemma 7.1 that any common prime divisor p of a triad satisfies $p|F_{2r}$ (in some instances, we even have $p|F_r$). When $r=1$, $F_{2r}=1$. □

7.2 The $g^2(-,-)$ Array

Consider the $g^2(-,-)$ array derived from generating sets of order 2 (Fig. 4). Hexagons of arbitrary size generally fail to satisfy the weak gcd property. Section 6 suggests we focus on those hexagons whose radii are divisible by $k=2$. We have the following result.

Lemma 7.3: Consider hexagons of radius $r=2a$, $a\in\mathbb{N}$, in the $g^2(-,-)$ array. If $a=1$, then the equal gcd property always holds (and in fact all gcd's of alternate triads equal 1). If $a\geq 2$, the weak gcd property always holds.

Proof: Consider a hexagon of radius $r=2$ centered about $g^2(n,t)$. We will show that each alternate triad has gcd equal to 1.

Consider first the case where both n and t are odd. Set $u=\lceil\frac{n}{k}\rceil$ and $v=\lceil\frac{t}{2}\rceil$. By Theorem 5.1, our alternate triads are:

$$g^2(n-2,t-2)=F_{u-2}F_{v-1}F_{u-v+1},$$
$$g^2(n,t+2)=F_{u+1}F_{v+1}F_{u-v},$$
$$g^2(n+2,t)=F_uF_vF_{u-v+2},$$

and

$$g^2(n-2,t)=F_{u-2}F_vF_{u-v},$$
$$g^2(n,t+2)=F_{u-1}F_{v-1}F_{u-v+2},$$
$$g^2(n+2,t+2)=F_uF_{v+1}F_{u-v+1}.$$

Let p be a common prime divisor for the first triad. By observations **A** and **B**, it is impossible for p to be a common divisor of any two of F_{u-2}, F_{u-1}, or F_u. It must be the case that p divides at least two of $F_{v-1}F_{u-v+1}$, $F_{v+1}F_{u-v}$, and F_vF_{u-v+2}. Again, noting **A** and **B**, this allows six possibilities:

i) $p|F_{v-1}$ and $p|F_{u-v}$ (and consequently $p|F_u$).

ii) $p|F_{u-v+1}$ and $p|F_{v+1}$ (and consequently $p|F_u$).

iii) $p|F_{v-1}$ and $p|F_{u-v+2}$ (and consequently $p|F_{u-1}$).

iv) $p|F_{u-v+1}$ and $p|F_v$ (and consequently $p|F_{u-1}$).

v) $p|F_{v+1}$ and $p|F_{u-v+2}$ (and consequently $p|F_{u-2}$).

vi) $p|F_{u-v}$ and $p|F_v$ (and consequently $p|F_{u-2}$).

Let F_m be the first Fibonacci number such that $p|F_m$, and consider case i). By **H** we have

$$\begin{aligned} v-1 &\equiv 0 \pmod{m}, \\ u-v &\equiv 0 \pmod{m}, \\ u &\equiv 0 \pmod{m}. \end{aligned}$$

Consequently $m=1$ and $p|F_m=1$.

Similarly, the remaining cases yield contradictions. Thus, the greatest common divisor of the first triad must be 1. Similarly for the second triad.

The same argument applies to the cases n even, and n odd, t even.

We will now establish the weak gcd property for hexagons of radius $r=2a$, $a\in\mathbb{N}$. Again, set $u=\lceil\frac{n}{k}\rceil$ and $v=\lceil\frac{t}{2}\rceil$ and consider the case n odd, t odd. The alternate triads are:

$$\begin{aligned} g^2(n-2a,t-2a) &= F_{u-1-a}F_{v-a}F_{u-v+1}, \\ g^2(n,t+2a) &= F_{u-1}F_{v+a}F_{u-v+1-a}, \\ g^2(n+2a,t) &= F_{u-1+a}F_vF_{u-v+1+a}, \end{aligned}$$

and

$$\begin{aligned} g^2(n-2a,t) &= F_{u-1-a}F_vF_{u-v+1-a}, \\ g^2(n,t-2a) &= F_{u-1}F_{v-a}F_{u-v+1+a}, \\ g^2(n+2a,t+2a) &= F_{u-1+a}F_{v+a}F_{u-v+1}. \end{aligned}$$

Let p be a common prime divisor of the first triad. There are 27 possibilities as to which Fibonacci factors it must divide. We must show that each scenario forces p to be a common divisor of the second triad. We will illustrate the four typical arguments used to demonstrate this. We leave the details of applying these arguments to the remaining 23 cases to the diligent reader.

Suppose $p|F_{u-1-a}$, $p|F_{v+a}$, and $p|F_{u-v+1+a}$. Then p is trivially a common divisor of the second triad.

Suppose $p|F_{u-1-a}$, $p|F_{u-1}$. Then, by **E**, $p|F_{u-1+a}$ and so p is a common divisor of the second triad.

Suppose $p|F_{u-1-a}$, $p|F_{v+a}$. Then, by **D**, $p|F_{2a}$ and, by **E**, $p|F_{v+a-2a}=F_{v-a}$ and so p is a common divisor of the second triad.

Suppose $p|F_{u-1+a}$, $p|F_{v+a}$, and $p|F_{u-v+1}$. Let F_m be the first Fibonacci number such that $p|F_m$. Then, by **H**,

$$\begin{aligned} u-a+1 &\equiv 0 \pmod{m}, \\ v+a &\equiv 0 \pmod{m}, \\ u-v+1 &\equiv 0 \pmod{m}. \end{aligned}$$

This is possible only if $m=1$ or $m=2$. But $p|F_m$ yields a contradiction. Therefore, this scenario cannot occur.

The remaining cases n even, and n odd, t even, are proven similarly. □

The following example shows that the equal gcd hexagonal property fails even for the case $a = 2$. Consider the hexagon of radius 4 centered about $g^2(44, 19)$. Then the alternate triads are:

$$g^2(40,15) = F_{20}F_8F_{13} = 6765 \cdot 21 \cdot 233,$$
$$g^2(44,23) = F_{22}F_{12}F_{11} = 17711 \cdot 144 \cdot 89,$$
$$g^2(48,19) = F_{24}F_{10}F_{15} = 46368 \cdot 55 \cdot 610,$$

with gcd $= 9$, and

$$g^2(40,19) = F_{20}F_{10}F_{11} = 6765 \cdot 55 \cdot 89,$$
$$g^2(44,15) = F_{22}F_8F_{15} = 17711 \cdot 21 \cdot 610,$$
$$g^2(48,23) = F_{24}F_{12}F_{13} = 46368 \cdot 144 \cdot 233,$$

with gcd $= 3$.

(Challenge for the reader: Prove that any common prime divisor p of an alternate triad from a hexagon of radius 4 in the $g^2(-,-)$ array must be a divisor of $F_8 = 21$. Consequently $p = 3$ or 7.)

7.3 The $g^k(-,-)$ Array, $k \geq 3$

In general, not even the weak gcd hexagonal property holds for $g^k(-,-)$, $k \geq 3$, arrays, even if the hexagon is of radius divisible by k. One can easily find examples to illustrate this. A simple one is the hexagon of radius 3 in the $g^3(-,-)$ array centered about $n = 14$, $t = 5$. Here the alternate triads are:

$$g^3(11,2) = F_4F_3F_1F_4 = 18,$$
$$g^3(14,8) = F_5F_4F_3F_3 = 60,$$
$$g^3(17,5) = F_6F_5F_2F_5 = 200,$$

with gcd $= 2$, and

$$g^3(13,5) = F_4F_3F_2F_3 = 12,$$
$$g^3(14,2) = F_5F_4F_1F_5 = 75,$$
$$g^3(17,8) = F_6F_5F_3F_4 = 240,$$

with gcd $= 3$.

This completes our analysis of the $g^k(-,-)$ arrays. We summarize our results in the following theorem.

Theorem 7.4: Concerning gcd hexagonal properties for hexagons of radius $r = ka$ in the array $g^k(-,-)$ (with $a, k \in \mathbb{N}$) we have the following:

1) For $k = 1$ and $k = 2$: The equal gcd property folds for $a = 1$. The weak gcd property holds for $a \geq 2$.

2) For $k \geq 3$: The weak gcd property fails.

As a final comment, we note that the equal lcm hexagonal property does not hold for the arrays $g^k(-,-)$.

REFERENCES

1. Shiro Ando & Daihachiro Sato. "A GCD Property on Pascal's Pyramid and the Corresponding LCM Property of the Modified Pascal Pyramid." In *Applications of Fibonacci Numbers* **3**:7-14. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1990.

2. Shiro Ando & Daihachiro Sato. "Translatable and Rotatable Configurations Which Give Equal Product, Equal GCD and Equal LCM Properties Simultaneously." In *Applications of Fibonacci Numbers* **3:**15-26. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1990.
3. Shiro Ando & Daihachiro Sato. "On the Proof of GCD and LCM Equalities Concerning the Generalized Binomial and Multinomial Coefficients." In *Applications of Fibonacci Numbers* **4:**9-16. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1991.
4. Shiro Ando & Daihachiro Sato. "A Necessary and Sufficient Condition that Rays of a Star Configuration on Pascal's Triangle Cover Its Center With Respect to GCD and LCM." In *Applications of Fibonacci Numbers* **5:**11-36. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1993.
5. Shiro Ando & Daihachiro Sato. "On the Minimal Center Coverings Stars With Respect to GCD in Pascal's Pyramid and Its Generalizations." In *Applications of Fibonacci Numbers* **5:**37-46. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1993.
6. R. P. Grimaldi. "Generating Sets and the Fibonacci Numbers." *Congressus Numerantium* **110** (1995):129-36.
7. H. W. Gould. "A New Greatest Common Divisor Property of the Binomial Coefficients." *The Fibonacci Quarterly* **10** (1972):579-84, 628.
8. A. P. Hillman & V. E. Hoggatt, Jr. "A Proof of Gould's Pascal Hexagon Conjecture." *The Fibonacci Quarterly* **10** (1972):565-68, 598.
9. V. E. Hoggatt, Jr., & G. L. Alexanderson. "A Property of Multinomial Coefficients." *The Fibonacci Quarterly* **9** (1971):351-56, 420-21.
10. V. E. Hoggatt, Jr., & W. Hansell. "The Hidden Hexagon Squares." *The Fibonacci Quarterly* **9** (1971):120, 133.
11. C. T. Long & Shiro Ando. "Two Generalizations of Gould's Star of David Theorem." In *Applications of Fibonacci Numbers* **4:**219-30. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1991.
12. C. T. Long & V. E. Hoggatt, Jr. "Sets of Binomial Coefficients and Equal Products." *The Fibonacci Quarterly* **12.1** (1974):71-79.
13. C. L. Moore. "More Hidden Hexagon Squares." *The Fibonacci Quarterly* **11** (1973):525.
14. D. Singmaster. "Notes on Binomial Coefficients: IV–Proof of a Conjecture of Gould on the GCD's of Two Triples of Binomial Coefficients." *The Fibonacci Quarterly* **11** (1973):282-284.
15. E. G. Strauss. "On the Greatest Common Divisor of Some Binomial Coefficients." *The Fibonacci Quarterly* **11** (1973):25-26.

AMS Classification Numbers: 11B39, 11B99

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